EI1120 Electric Circuit Analysis (CENMI), Homework 4/4.

For presentation on 2013-03-01, 11–12 (after examples-session 09–11).
In English, by request of several students.
Please treat only 1a–1e, 2a–2e and 3a–3d as the core part that you should try to answer for the homework/redovisning. In that case, the work-load should be similar to homeworks last year or in the Elektro and IT programmes (you do not, of course, even have to answer correctly all of this subset). Other parts are only for those who have time and interest. However, it's recommended that everyone looks at all the questions and checks with the solutions when they're published on Friday. The tasks here will also provide some revision of simple nodal analysis, equivalent sources and rms values.

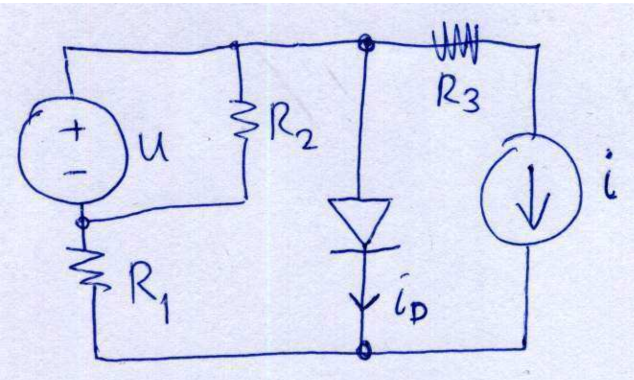
Nathaniel Taylor, Feb 2013

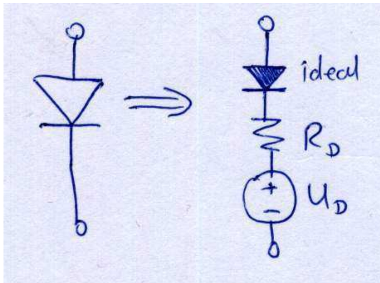
---

1) Diodes (with node analysis and equivalent source)

The diagram below left shows a circuit consisting of an nonideal diode and some other, *linear*, components. All marked quantities are known *except* i_D.

On the right is shown one of many models that can be used to model an nonideal diode. In this model a series voltage source models the 'forward drop' as a constant u_D (which can be about 0.6 V for a silicon diode), and a constant series resistance R_D models a further current-dependent voltage-drop; an ideal diode then ensures that no current can flow in the reverse direction. An ideal diode conducts like a short circuit if the current tries to go in the direction of the arrow, and behaves like an open-circuit otherwise. The u_D and R_D are thus only relevant to the external circuit (beyond the nonideal diode's terminals) if the ideal diode in the model is in its conducting state.





Remember: one way to analyse the case of one ideal diode within an otherwise linear circuit is to:

> *Assume* either that the diode is a short-circuit or an open-circuit.
>
> Calculate the current through the short-circuit, or the voltage across the open-circuit (depending on which state you assumed).
>
> If that current or voltage is in the forward direction for the diode, then the diode would be forward-biased (conducting: a short-circuit); otherwise it would be reverse-biased (non-conducting: open circuit).
>
> If this result is what you'd initially assumed, then you already have the solution to the circuit. For example, if you assumed a short-circuit, and then solved and found a current through the ideal diode in the forward direction, then you have correctly calculated the circuit's current and can assume the voltage across the diode is zero (because it is forward biased and ideal). If instead your initial assumption has turned out wrong, you need to recalculate with the diode in the opposite state. (Clearly, it's an advantage to take a little time on estimating the right state from the start!)

We analyse now the circuit shown on the left.

a) We want to calculate the unknown current marked in the circuit as i_D. The nonideal diode will be modelled as shown on the right. Draw the resulting circuit, with any simplifications that can be made (things that do not have an effect on i_D).

b) Assume the diode is conducting (so that the ideal diode is a short-circuit). Use nodal analysis to express the current i_D in terms of the known variables. Note: this is a very simple case of 'nodal analysis': you should only need to use KCL at one node. The suitably simplified circuit has only two significant nodes (one can be treated as ground) as well as three 'trivial' nodes (where only two components join, so there is the same current in both).

c) The numerical values for the components (note: all sources are dc) are: $u = 10\,\mathrm{V}$, $i = 3\,\mathrm{A}$, $u_\mathrm{D} = 0.6\,\mathrm{V}$, $R_\mathrm{D} = 0.2\,\Omega$, $R_1 = 5\,\Omega$, $R_2 = 26.43\mathrm{e}^{0.4321\pi^{2.22}}\,\mathrm{m\Omega}$, and $R_3 = 0.123456789\,\mathrm{M\Omega}$. Given these values, would the ideal diode really have been behaving like a short circuit (forward-biased), as assumed in our calculation in b)?
Note: if your simplified circuit still contains R_2 or R_3, please check your solution to part a), then recalculate b) – these components are *both irrelevant* (why?), hence the silly values given for them!

d) Assume now that the ideal diode is behaving like an open circuit (reverse-biased). Again, use nodal analysis in a simple way, but this time determine the voltage across the open-circuited ideal diode: express this symbolically in terms of known variables.

e) Put in the numerical values to calculate the voltage across the ideal diode. Would the ideal diode really have been behaving like an open circuit (reverse-biased)? We hope the answer to this is the opposite of the answer to c)!
(The possible exception would be if the values of u, R_1, i, u_D and R_D are exactly at the critical point between forward and reverse bias of the ideal diode: then one can't really say it's an open circuit or short circuit; it has properties of both, as its voltage and current are both zero.)

f) Now we'll use a different approach as an alternative to parts b)–e). Using your circuit of part a), replace everything *apart from the ideal diode* with an equivalent source (Thevenin or Norton), with the source's resistance, and voltage or current, expressed in the known variables.

g) Use values for u, i, etc. given in part c), to find the actual source and resistance value for your equivalent source. Draw this equivalent source connected to an ideal diode. Can you see immediately the solution to how the ideal diode is biased (forward or reverse) and what the current through it and voltage across it are?

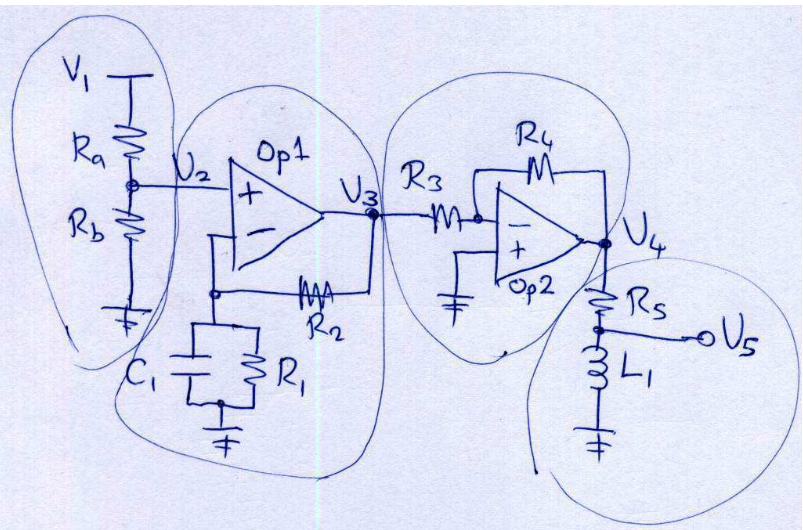
h) Write a general expression for the relation of u, i, R_1, u_D at the critical point when the diode is between the forward and reverse states (see discussion in part e)). Why isn't R_D important?

i) Do a dimensional analysis of the expression derived in h).

j) We've used simple node analysis to calculate i_D, and then an alternative method of a Thevenin or Norton equivalent. Another common method that we have seen in this course is superposition. This seems a reasonable choice when we want to calculate a current due to several independent sources. Could we have used superposition to solve this problem? Are there any 'reservations' (anmärkningar: speciellvillkor) for superposition working here. Please explain: you're welcome to give an example by using superposition on the problem.

---

2) Filters (including opamps and Bode plots)



The circuit on the left shows two opamps together with seven resistors, a capacitor and an inductor. It forms a filter from input voltage V_1 to output voltage V_5, both expressed as potentials.

Note that there are several 'earth-points' (or 'grounds'): these points can all be assumed to be connected together (one sometimes omits the wires between them to simplify the diagram).

Between the marked potentials $V_1, V_2, \ldots V_5$ there are 4 'stages', which have been marked by drawing a ring around each one.

In the particular circuit shown, these stages can be treated independently, then their network functions can be chained together (multiplied) to give the network function of the entire circuit. This convenient situation is *not* always true when an output of one part of a circuit feeds another part: the 'other part' might change the output value of the first part due to its own impedance ('loading', like the classic problem of a voltmeter or ammeter with finite resistance).

a) For each of these four stages, derive the network function.
In other words, express the relations: $\frac{V_2}{V_1}$, $\frac{V_3}{V_2}$, $\frac{V_4}{V_3}$ and $\frac{V_5}{V_4}$. Two of these stages have a constant gain at all frequencies. One of these is a divider (it must reduce the voltage): which is this? The other is a general

amplifier (it could reduce, increase, or preserve the voltage): which stage is this? Another stage is a common type of simple filter: which stage is this?, and what type (high-pass, band-pass, low-pass, band-stop)?

b) Multiply these expressions together to find the total network function $\frac{V_5}{V_1}$. (Note the convenience compared to dealing with differential equations in the time-domain!)

c) Do careful dimensional analysis of the solution to b). Work in the same order as if you were calculating the answer with numerical values of the variables, i.e. in the expression $a + (b + c/d)^2$ you would start with c/d, check that this dimension is the same as of b, then check that the square of this dimension is the same as the dimension of a. It is very useful to *learn* the common things like $\omega C = [\mathrm{A}]/[\mathrm{V}]$ (admittance), etc.

d) Show that the network function in b) can be written in the canonical form

$$\frac{V_5}{V_1} = k\frac{-\mathrm{j}(\omega/\omega_1)\,(1+\mathrm{j}\omega/\omega_2)}{(1+\mathrm{j}\omega/\omega_1)},$$

where k, ω_1 and ω_2 are positive real constants. What are the values of k, ω_1 and ω_2, in terms of the known component-values?

e) Plot the Bode plots of magnitude (dB) and phase (degrees) for the network function of part d), in terms of k, ω_1 and ω_2 (i.e. don't bother with all the R_1 etc. values from part b)). Assume $\omega_1 < \omega_2$, and plot a range that includes the interval between these frequencies and extends a similar interval on either side. You need not be accurate about the dB level, but just about which parts are flat and which are sloping at (what slope? $+20\,dB$, -40,dB, etc.); to know the exact level you would need component values (you can use the following if you really want values `Ra=2000; Rb=1000; C1=212e-9; R1=3000; R2=1000; R3=2700; R4=10000; R5=10; L1=0.16;`, which would give nice corner-frequencies when expressed in hertz).

f) Consider now the case where C_1 and R_1 are removed (open-circuit). Opamp 'Op1' then becomes a buffer amplifier, with gain of 1, for any finite value of R_2 (make sure you understand why). Write the new network function $\frac{V_5}{V_1}$. This should be very easy, as you already have done all the hard work in part a).

g) Electronic designers probably want to save on parts and thus on money. In the situation of part e), the combination of Op1 and R_2 is giving the same voltage-gain as a piece of wire. Perhaps we can save money by not having Op1 present at all in the circuit, if we just connect a wire between points V_2 and V_3.... Can we do this, without changing the network function compared to the situation described in part e)? Justify your answer.

h) Calculate the network function $\frac{V_5}{V_1}$ for the case described in part g) (where points V_2 and V_3 are shorted). This can't be done entirely by the solutions found in part a), but it is quite easy because of how Op2 'isolates' the two sides of the circuit from each other (there's a 'virtual ground' on its inverting input) so that the current in through R_3 does not depend on R_4, R_5 or L_1.

i) What condition on R_b and R_3 would make the solutions for f) and h) be approximately the same?

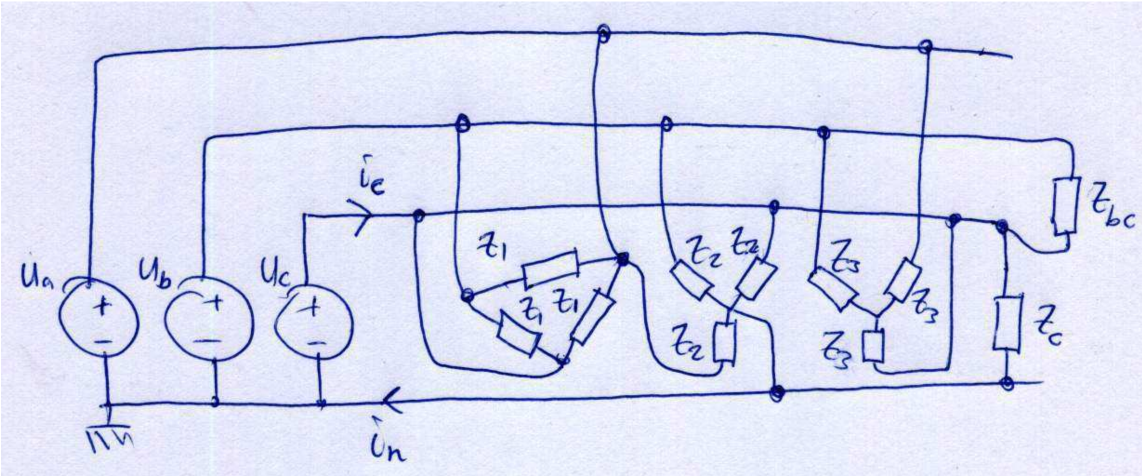
You might like to know that the above circuit was made up entirely as an example, with the desire of showing the importance of loading (input impedance), and providing a filter built of sub-filters. "We make no claim that this would be chosen as a filter design in any practical application!"

---

3) Three-phase systems.

The crudely sketched diagram below shows a 3-phase 4-wire supply system modelled as three ideal voltage-sources. These give phase-voltages that can be represented as phasors such that $u_\mathrm{a} = U\angle 0$, $u_\mathrm{b} = U\angle{-2\pi/3}$, and $u_\mathrm{c} = U\angle{-4\pi/3}$, where U is the rms (effective) value of the sinusoidal voltage.

Note that there are three *balanced* 3-phase loads: load 1 has impedances Z_1 in Δ (delta) connection; load 2 has impedances Z_2 in Y ('wye', or 'star') connection with the centre ('star-point') connected to the supply's neutral conductor; load 3 has impedances Z_3 star-connected with no neutral connection.



The only other two components shown are a single impedance Z_{c} from phase 'c' to neutral (the earthed [or 'grounded'] conductor at the bottom), and another impedance Z_{bc} between phases b and c.

a) For each balanced load (1,2,3) write an expression for the load's total (all three impedances) complex power. Then write expressions for the complex power into the other loads Z_{c} and Z_{bc}.

b) What is the current i_{c}? (Express it as a phasor, in terms of known impedances and the rms phase-voltage magnitude U.

c) What is the current i_{n}?

d) [Very simple reminder of rms and 3-phase basics.] Given that the rms phase-voltage from the source is U, give the following (as absolute values, not caring about phase-angle),
i) what is the peak phase-voltage?
ii) what is the rms line voltage (voltage between two phases)?
iii) what is the peak of the line voltage?

---

4) More diodes. **Not a real "question".**

For anyone interested in some applications of diodes, 'rectifiers' (likriktare) are a good example. Wikipedia has a good description: `http://en.wikipedia.org/wiki/Rectifier`. This moves from a simple single diode (like our demonstration circuit used in transients lectures with two lamps, two switches, a big capacitor and a diode), then a 4-diode 'bridge' circuit with better properties of smoothness, and then on to 3-phase rectifiers with 6 diodes or more.

Many small electronic devices such as desktop computers, monitors, chargers, etc., have a very simple power-input stage where a bridge-type rectifier keeps a capacitor charged at close to the peak voltage of the ac power. Then more clever electronics (a bit like the motor controller we saw in a guest lecture) quickly switch this dc to generate a desired output voltage. There is usually a high-frequency transformer in the middle, to prevent a direct metallic connection between the power network and the low-voltage output: remember, from HU3, transformers can be smaller at higher frequency.

A disadvantage of the simple diode-based rectifier, when feeding a capacitor to give a quite steady dc output, is that almost all the current flows around the time of the peak of the ac voltage (which is the only time when the ac input is greater than the capacitor voltage). This causes the input current to have harmonics (higher fourier-series components: not just one pure sinusoid): these can disturb other equipment, and result in a higher rms value compared to a pure sinusoid deliering the same amount of useful power (therefore, higher losses in the wires). For larger computers, the current is high enough that rules require the manufacturers to have a cleverer input stage, which uses quick switching (kilohertz) to force the input current in an inductor to follow approxiately a sinusoidal shape. Some examples of input currents to computing equipment can be seen here: `http://gnu.ets.kth.se/~nt/elecpow/it_power` (I measured these several years ago). Compare for example the computer 'backup' (a basic one) with 'penguin' (cleverer; a server).

More and more devices are being based on power electronics: another example is the fluorescent low-energy lamp that has largely replaced incandescent lamps (glödlampor). Large amounts of non-sinusoidal current have a bad effect on the local power network, with regard to heating, neutral-currents, and disturbance to other electrical equipment (including our attempted experiments in the high-voltage laboratory!); some input-current waveforms of small cheap lamps can be seen here `http://gnu.ets.kth.se/~nt/elecpow/le_lamps`.

On the subject of car alternators (generators), the following pages give good pictures and diagrams:
`http://www.allaboutcircuits.com/vol_6/chpt_4/8.html` and
`http://www.carparts.com/classroom/charging.htm`.

Note how the car alternator is in fact a 3-phase star-connected machine, with no neutral; it then rectifies the current through 6 power-diodes (and a further 3 that connect to the warning circuit and ignition switch).

Look at the 6 diodes connecting the 3-phase coils to the battery. Why doesn't the battery discharge into the alternator coils when the alternator isn't rotating (isn't generating its own voltage)? What would happen, even with the engine not running (no rotation) if someone accidentally connected a car's battery the wrong way round (+ and − poles reversed)?

The motors in some electric or hybrid cars are a bit like this alternator in reverse: power-electronic switches convert dc from a battery into 3-phase ac, which is fed into a 3-phase winding in a motor. Motors and generators are fundamentally 'reversible': one can function as the other, although not necessarily with optimal properties! There are many types of motor: not all electric vehicles have motors that use ac. Some have dc fed to the coils, some have ac; some use permanent magnets on the rotating part, some feed dc to the rotating part to give a controllable magnetic field, some induce currents in the rotating part via mutual inductance, etc.

Compiled: February 27, 2013.
Important changes to Q2 (esp 2a,2d) 26 February 2013.
First version: 24 February, 2013.

EI1120 Electric Circuit Analysis (CENMI), SOLUTIONS to Homework 4/4.

Presentation of homework tasks on 2013-03-01, 11–12 (after examples-session 09–11).
In English, by request of several students.
Only 1a–1e, 2a–2e and 3a–3d were presented as the core part that should be answered reasonably well for the homework/redovisning.
It is however recommended that everyone looks at all the questions and checks with the solutions now that they're published.

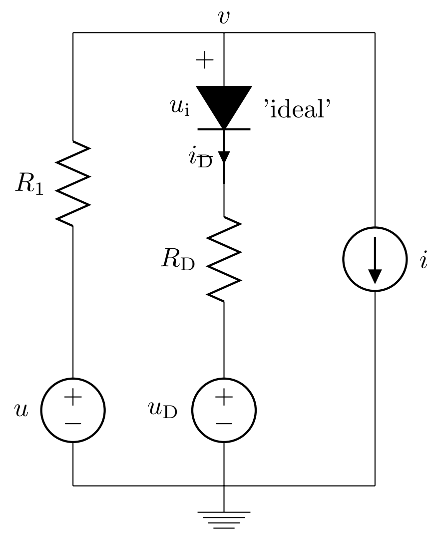
Nathaniel Taylor, Feb 2013

---

1) Diodes (with node analysis and equivalent source)

a) The resulting circuit, including the nonideal-diode model and some simplifications (that do not have an effect on i_D) is shown on the right. The potential v is *not* known; it is there for use in later questions!

As stated in question c), the resistors R_2 and R_3 were irrelevant for anything in the circuit apart from these resistors themselves (which we didn't care about) and the sources they were directly in parallel or series with (respectively). R_2 would have caused a further current in the voltage source u, but the terminals connecting to the rest of the circuit still behave like a voltage source. R_3 would have caused a further voltage drop in its branch, which would have required a different voltage across the current source i; but the branch would still behave like a current source. The irrelevance of components parallel with a voltage source or series with a current souce has often been seen in this course! Just remember that it's *only true* when we analyse the rest of the circuit and don't care about finding all the details (i, u, P) for the source or the parallel/series-coupled component.



b) In the circuit of a), assume the ideal diode is conducting (short-circuit). Use nodal analysis to obtain i_D.

There are three parallel branches between the top node v and the bottom node (which we will choose as the reference node: ground). The node between u and R_1, and the nodes between the ideal diode and R_D, and R_D and u_D, are 'trivial': they join just two component-terminals, so the current is the same in both, and it is easy to express the entire branch current without needing to define the potentials of the trivial nodes.

In the diode branch (middle branch), the current is

$$i_D = \frac{v - u_D}{R_D}.$$

Using KCL for the 'top node' (above the diode, marked as unknown potential v), we can express three currents going out from this node: $(v-u)/R_1$ in the left branch, and i in the right branch,

$$0 = \frac{v-u}{R_1} + \frac{v-u_D}{R_D} + i,$$

whence

$$v\left(\frac{1}{R_1} + \frac{1}{R_D}\right) = \frac{u}{R_1} + \frac{u_D}{R_D} - i$$

$$\therefore v = \frac{R_1 R_D}{R_1 + R_D}\left(\frac{u}{R_1} + \frac{u_D}{R_D} - i\right).$$

which is probably best left in that form.[1]

Now we can calculate i_D from this potential:

$$i_D = \frac{v - u_D}{R_D} = \frac{u_1 - u_D - iR_1}{R_1 + R_D}$$

c) Given these values: $u = 10\,\text{V}$, $R_1 = 5\,\Omega$, $u_D = 0.6\,\text{V}$, $R_D = 0.2\,\Omega$, $i = 3\,\text{A}$, the numerical value of the current i_D (through the short-circuit that replaced the ideal diode) is conveniently calculated by computer:

[1] This form of the equation is nice: the term on the right is the short-circuit current of all three branches if one shorted the node v to ground, and the term on the left is the total impedance of the three branches between v and ground. This rule for the voltage of several parallel branches of sources and loads is called 'Millman's theorem' or the 'parallel generator theorem'.

```
u=10; R1=5; ud=0.6; Rd=0.2; i=3;
v = (R1*Rd/(R1+Rd)) * (u/R1 + ud/Rd - i)
   --> v =  0.385
id = (v-ud)/Rd
   --> id = -1.077
```

i.e. a negative current (a current in the reverse direction), which would not really be allowed by the diode. That's a pity: it seems someone made the *wrong* assumption from the start, possibly with the purpose of protracting this question all the way to part 'j'....

d) So, now we assume (with confidence) that the ideal diode is actually behaving like an *open* circuit (reverse-biased). We use node analysis again, to find the open-circuit voltage on the diode.

Open-circuit means $i_\mathrm{D} = 0$, which means zero voltage across R_D: therefore, if we find v, we can find the voltage across the the open-circuited ideal diode as $v - u_\mathrm{D}$. The nodal analysis is extremely simple in this case, as there are only 2 branches.

$$0 = \frac{v-u}{R_1} + i$$

$$\therefore v = R_1 \left(\frac{u}{R_1} - i \right),$$

and the simplified version of this could have been seen immediately by considering the left branch having a constant current i forced up it by the current source (as the middle branch is open-circuit), thus

$$v = u - iR_1.$$

The voltage across the ideal diode is therefore

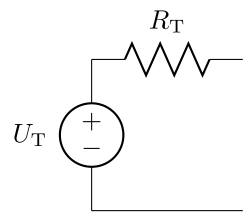
$$u_\mathrm{i} = u - iR_1 - u_\mathrm{D}.$$

e) Put in the numerical values to calculate the voltage across the ideal diode.

```
u=10; R1=5; ud=0.6; Rd=0.2; i=3;
ui = u - i*R1 - ud
   -->  -5.6
```

i.e., $-5.6\,\mathrm{V}$. *Yes,* the ideal diode really would have been behaving like an open circuit (reverse-biased).

f) For the circuit of part a), replace everything *apart from the ideal diode* with an equivalent source (Thevenin or Norton). So: we have a choice of either type of equivalent source, and for whichever we choose we have several possible ways of determining the equivalent components (U,R) or (I,R).[2]



I'll choose a Thevenin source. Its Thevenin voltage will be the open-circuit voltage that we calculated in part 'd)':

$$U_\mathrm{T} = u - iR_1 - u_\mathrm{D},$$

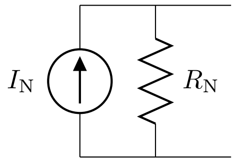
(which we know to be $-5.6\,\mathrm{V}$ with our component values). Its series resistance is the resistance if we set all the sources u, i and u_D to zero then measure resistance between the open-circuited terminals where the ideal diode was. This is very easy: the voltage sources become short-circuits and the current source is an open circuit, which means we have just one loop of resistors R_1 and R_D in series:

$$R_\mathrm{T} = R_1 + R_\mathrm{D},$$

which is $5.2\,\Omega$ in our case.

[2] Remember: for either source we need to determine two things: the internal impedance (which is the *same* for Thevenin or Norton) and the source strength (voltage source for Thevenin, current source for Norton). In either case, we can do this by determining one point in the i/v relation of the circuit, and then determining the internal impedance, *or* we can determine two points in the i/v relation; as all linear circuits have a straight line relation of i and v, two points are sufficient to determine the slope and offset (classic straight-line graph $y = mx + a$, with unknown slope m and offset a). Often the most convenient two points to calculate are the short-circuit voltage and open-circuit current: these are relatively simple to measure in practical cases (yes, people really do make measurements on circuits to find their equivalents), and they often result in simpler circuits to analyse in theoretical cases (because short-circuiting reduces the number of nodes, and open-circuiting guarantees zero volt-drop across series output resistors). Without dependent sources it is often quicker to calculate just the required voltage or current for the equivalent circuit, then to calculate the equivalent resistance directly by setting independent sources to zero and calculating the resistance between terminals. However, with dependent sources present this method can be impossible (in which case one might calculate internal resistance from the terminal voltage when one forces a known non-zero current through the circuit). Then it is sometimes easier just to calculate the short-circuit current and open-circuit voltage.

It would be similarly easy to derive a Norton source, as we also know the short-circuit current thanks to part 'b)': this would be the Norton current-source current.



$$I_\mathrm{N} = \frac{v - u_\mathrm{D}}{R_\mathrm{D}} = \frac{u_1 - u_\mathrm{D} - iR_1}{R_1 + R_\mathrm{D}}$$

(which we know to be about $-1.08\,\mathrm{A}$). Its *parallel* resistance is, as always, exactly the same as the series resistance of an equivalent *Thevenin* source.

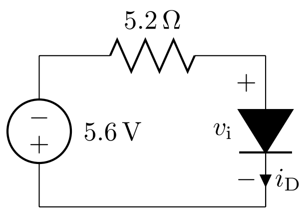
$$R_\mathrm{N} = R_\mathrm{T} = R_1 + R_\mathrm{D}.$$

In the above cases (Thevenin, Norton) the top terminal is the one corresponding to the upper terminal (+) on the ideal diode.[3]

Note that we could also have derived the resistance as the ratio of the open-circuit voltage to the short-circuit current at the terminals of the source: this was easy in our case as we already had calculated both of these values. Hence, we hope to find that $(-5.60/-1.077)$ [V/A] is equal to 5.2 [Ω] … and indeed it is. Doing this symbolically can be "left as an exercise for the reader".

g) Values for u, i, etc. have already been inserted in the equations in part f), to give us a Thevenin voltage of $-5.6\,\mathrm{V}$ or a Norton current of $-1.08\,\mathrm{A}$.

If we draw (right) an equivalent source with the ideal diode connected, the result is equivalent to the circuit in part a), seen from the diode.
This way, we see immediately that the ideal diode is reverse biased, and therefore that no current will flow, and that the voltage v_i will therefore be $-5.6\,\mathrm{V}$.



h) The critical point when the diode is between the forward and reverse states is when $U_\mathrm{T} = 0$ or $I_\mathrm{N} = 0$. From the expression for the Thevenin voltage, this requires

$$U_\mathrm{T} = 0 = u - iR_1 - u_\mathrm{D}.$$

i) Do a dimensional analysis of the expression derived in h). Easy: iR_1 has dimensions [A][Ω] which is [V]. Everything else is already dimension [V], and is just addition or subtraction. Thus: dimensionally correct.

j) Could we have used superposition to solve this problem?

Parts b) and d) *could* have been solved by superposition. We could have treated each of the three sources separately, or some combination such as u and u_D together, then i separately: anything like that works as long as every independent source has been considered exactly once. We are able to use superposition because the circuit is linear: it is linear because we have replaced the diode (which is very nonlinear) with a short-circuit or open-circuit (which can be seen as a linear component: a zero voltage-source or current-source!). Superposition in fact gives us a very nice quick solution.

However, we could have used superposition wrongly, due to the presence of a nonlinear component. It would be wrong to decide the actual voltage and current for the ideal diode 'caused' by each source separately, then to add them. For example: First take the two voltage sources together, with the current source set to zero (open-circuit); then we find the diode to be forward biased, from which we find a forward current of $(u - u_\mathrm{D})\,/\,(R_1 + R_\mathrm{D})$ and a zero voltage; Then, take just the current source, from which we find that the diode is *reverse* biased with voltage $-iR_1$ and therefore has zero current. If we add these results together (the usual final step of superposition) we find that the ideal diode has a voltage $0 - iR_1$ and current $(u - u_\mathrm{D})\,/\,(R_1 + R_\mathrm{D}) + 0$, suggesting it's reverse and forward biased at the same time. That's ridiculous, and it's because we've tried superposing solutions on the nonlinear component, and the nonlinear component had different states in the two superposed cases. We must do the superposition only on the linear replacement circuit (short-circuit or open-circuit), not the diode.

[3] The Thevenin and Norton sources have been drawn in a 'standard' way, with current going up and voltage positive at the top: this means that we get negative values for the current and voltage, since in our case it is the top terminal of the diode that actually has a lower potential. We could instead have removed the negative sign and made the sources point the opposite way.

2) Filters (including opamps and Bode plots)

a) For each of these four stages, we derive the network function.

Two of these stages have a constant gain at all frequencies – these are the ones without L or C components. The first stage is a divider (it must reduce the voltage):

$$\frac{V_2}{V_1} = \frac{R_\mathrm{b}}{R_\mathrm{a} + R_\mathrm{b}}.$$

It is *only because* the output of the divider goes into an opamp input with nothing else connected (therefore infinite input impedance in our simple opamp model) that we can directly use the voltage divider equation! If we had an inverting amplifier, there would be a finite input impedance which would 'load' the divider (i.e. put a further impedance parallel with R_b) and the divider equation would therefore have to be written differently.

The third stage also has constant gain: it is a general amplifier (it could reduce, increase, or preserve the voltage). Op2 has its non-inverting input grounded. The inverting input is therefore assumed also to be held at zero potential (because there is negative feedback through R_4). KCL at the inverting input gives us $(V_4 - 0)/R_4 + (V_3 - 0)/R_3 = 0$ (as no current goes into the opamp), which means that

$$\frac{V_4}{V_3} = \frac{-R_4}{R_3}.$$

The other two stages are filters. The second stage, around Op1, has a divider connecting the output potential V_3 to the inverting input; the inverting input should have equal potential to the non-inverting input, which is at V_2. From voltage division, the divider ratio is $V_2/V_3 = Z/(R_2 + Z)$ where $Z = 1/(\mathrm{j}\omega C_1 + 1/R_1)$. We want, however, the reciprocal, V_3/V_2, to give the filter's network function instead of the divider's network function (yes, that's the step that I forgot on my tenth time playing with different components in this circuit, hence the incorrect question in the first version of this homework!). Therefore, after some manipulation,

$$\frac{V_3}{V_2} = \frac{R_1 + R_2}{R_1}\left(1 + \mathrm{j}\omega C_1 \frac{R_1 R_2}{R_1 + R_2}\right).$$

The final stage is again easily calculated as a voltage divider, *provided that no significant current is taken from the output terminal*,

$$\frac{V_5}{V_4} = \frac{\mathrm{j}\omega L_1}{R_5 + \mathrm{j}\omega L_1} = \frac{\mathrm{j}\omega \frac{L_1}{R_5}}{1 + \mathrm{j}\omega \frac{L_1}{R_5}}.$$

b) We multiply the above four expressions together to find the total network function $\frac{V_5}{V_1}$.

$$\frac{V_5}{V_1} = \frac{R_\mathrm{b}}{R_\mathrm{a} + R_\mathrm{b}} \cdot \frac{R_1 + R_2}{R_1} \cdot \frac{-R_4}{R_3} \cdot \frac{\mathrm{j}\omega \frac{L_1}{R_5}\left(1 + \mathrm{j}\omega C_1 \frac{R_1 R_2}{R_1 + R_2}\right)}{1 + \mathrm{j}\omega \frac{L_1}{R_5}}$$

c) Dimensional analysis of the solution to b). Seeing that CR and L/R are time-constants (dimension [s]) and angular frequeny has dimension $[\mathrm{s}]^{-1}$, the solution b) has dimensions:

$$\frac{[\mathrm{V}]}{[\mathrm{V}]} = \frac{[\Omega]}{[\Omega]}\frac{[\Omega]}{[\Omega]}\frac{[\Omega]}{[\Omega]}\frac{[\mathrm{s}]^{-1}[\mathrm{s}]\left(1 + [\mathrm{s}]^{-1}[\mathrm{s}]\right)}{1 + [\mathrm{s}]^{-1}[\mathrm{s}]},$$

which clearly cancel in every term, leaving each small part of the equation dimensionless. A more pleasing analysis could be given by showing several steps of reduction of the problem.

d) The network function in b) can be written as,

$$\frac{V_5}{V_1} = k\frac{-\mathrm{j}\left(\omega/\omega_1\right)\left(1 + \mathrm{j}\omega/\omega_2\right)}{\left(1 + \mathrm{j}\omega/\omega_1\right)},$$

where $k = R_\mathrm{b}\left(R_1 + R_2\right) R_4/\left(\left(R_\mathrm{a} + R_\mathrm{b}\right) R_1 R_3\right)$, $\omega_1 = R_5/L_1$, and $\omega_2 = \frac{R1+R2}{C_1 R_1 R_2}$.

e) Plot the Bode plots of magnitude (dB) and phase (degrees) for the network function of part d), in terms of k, ω_1 and ω_2 (i.e. don't bother with all the R_1 etc. values from part b)). Assume $\omega_1 < \omega_2$.

Let us take the suggested numerical values, and put them into Octave/Matlab:

```
Ra = 2000;    Rb = 1000;
C1 = 212e-9; R1 = 3000;    R2 = 1000;
R3 = 2700;    R4 = 10000;
R5 = 10;      L1 = 0.16;.
```

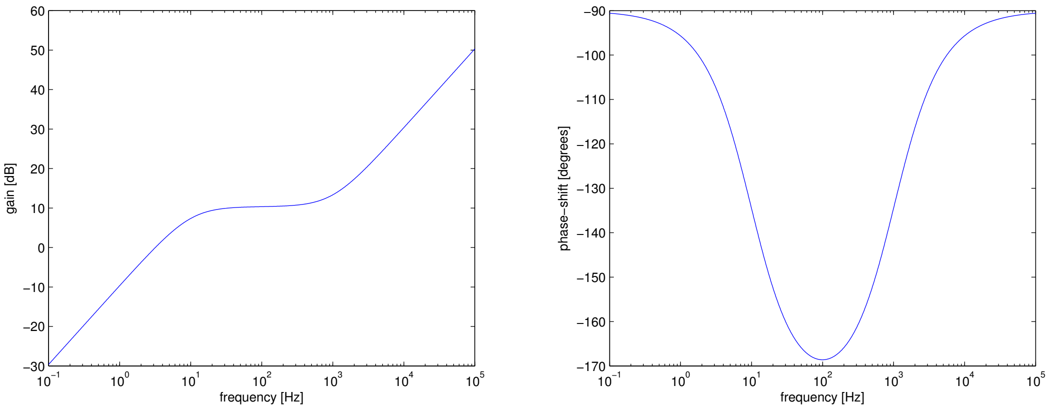
This gives ω_1 about 10 Hz (62 radian/s) and ω_2 about 1 kHz; the factor k is about 3.3.

```
k= Ra*R4*(R1+R2) / ( (Ra+Rb)*R1*R3 )
w1 = R5/L1
w2 = (R1+R2)/(C1*R1*R2)
```

We can now generate a list of frequencies, and calculate the network function for each one. Note the 'element-wise' multiplication and division, using a dot before the operator (`.*`, `./`).[4] If you want to understand more about these functions in GNU Octave or Matlab, use the help function: e.g. `help logspace`.

```
f = logspace(-1, 5, 2048);
w = 2*pi*f;
H = k * (-1j*w/w1) .* (1+1j*w/w2) ./ (1+1j*w/w1);
dB = 20*log10(abs(H));
ph = angle(H);

figure(1);
subplot(1,2,1);   semilogx(f, dB, 'b');
xlabel('frequency [Hz]'); ylabel('gain [dB]');
subplot(1,2,2);   semilogx(f, ph*180/pi);
xlabel('frequency [Hz]');  ylabel('phase-shift [degrees]');
```



If you sketch this bode-plot without actual numbers, the important points are that:

> The slope is +20 dB/decade at the lowest frequencies: at these frequencies $\omega \ll \omega_1$ and therefore the terms $(1 + \omega/\omega_x)$ are approximately 1, leaving just the term $-k\mathrm{j}\omega/\omega_1$, hence $|H| \propto \omega$.
>
> The slope is also +20 dB/decade at the highest frequencies, where the terms $(1+\omega/\omega_x)$ together become approximately ω_1/ω_2, so the solution is $-k\mathrm{j}\omega/\omega_2$, hence again $|H| \propto \omega$.
>
> Between the corner frequencies of the two $(1 + \omega/\omega_x)$ terms, there is a region of almost flat response (slope of zero).
>
> The phase can be seen by the same reasoning: at very high or low frequencies the combination of the two $(1 + \mathrm{j}\omega/\omega_x)$ terms gives zero phase, so the total phase is the $-90°$ due to the $-\mathrm{j}\omega/\omega_1$ term; between ω_1 and ω_2 there is a point where the denominator gives significant extra phase-lag, before the term in the numerator also starts being dominated by the $\mathrm{j}\omega/\omega_2$ part.

f) Consider now the case where C_1 and R_1 are removed (open-circuit), so that opamp 'Op1' becomes a buffer amplifier (gain of 1, i.e. 0dB). The new network function $\frac{V_5}{V_1}$ is found from b), by removing the parts corresponding to the second stage:

$$\frac{V_5}{V_1} = \frac{R_\mathrm{b}}{R_\mathrm{a} + R_\mathrm{b}} \cdot \frac{-R_4}{R_3} \cdot \frac{\mathrm{j}\omega\frac{L_1}{R_5}}{1 + \mathrm{j}\omega\frac{L_1}{R_5}}$$

[4] The dot makes it so that the result of multiplying two $N \times 1$ vectors `a` and `b` is another $N \times 1$ vector `c` such that `c(n) = a(n)*b(n)` for each n from 1 to N; if not using the dot, this would be an error, as matrix-based multiplication normally expects to get input like an $N \times 1$ times $1 \times N$ (giving an $N \times N$ matrix) or $1 \times N$ times $N \times 1$ (giving a 1×1 scalar) result. The element-wise operations are useful for writing 'vectorized' program that applies an equation to every element in a vector or matrix (every frequency value in our case) without having to write a loop.

g) If we just connect a wire between points V_2 and V_3, would we get the same result as in f)?

No! The input to the third stage looks like a resistor R_3 going to ground (virtual ground). This will draw current from the source that feeds it. If we connect directly to the divider of R_a and R_b then we load the divider: it becomes really a divider where the bottom impedance is R_b and R_3 in parallel instead of just R_b. Its ratio therefore changes, so the total network function changes. By having the buffer amplifier, this loading does not happen.

h) We now calculate the network function $\frac{V_5}{V_1}$ for the case of part g). The only difference needed compared to part f) is that the first stage divider has to be replaced with the divider described in g).

Thus the term $\frac{R_\mathrm{b}}{R_\mathrm{a}+R_\mathrm{b}}$ in f) becomes

$$\frac{\frac{R_\mathrm{b}R_3}{R_\mathrm{b}+R_3}}{R_\mathrm{a}+\frac{R_\mathrm{b}R_3}{R_\mathrm{b}+R_3}},$$

so the total network function can be written

$$\frac{V_5}{V_1} = \frac{R_\mathrm{b}R_3}{R_\mathrm{a}\left(R_\mathrm{b}+R_3\right)+R_\mathrm{b}R_3} \cdot \frac{-R_4}{R_3} \cdot \frac{\mathrm{j}\omega\frac{L_1}{R_5}}{1+\mathrm{j}\omega\frac{L_1}{R_5}}.$$

i) The solutions for f) and h) would be approximately the same if $R_3 \gg R_\mathrm{b}$. This can be seen by 'circuit reasoning' if one considers that the current drawn by R_3 must be much less than the current in the divider, if it won't influence the divider voltage strongly by it 'loading'. Alternatively we can work mathematically, seeing that the term $(R_\mathrm{b}+R_3)$ in 'f)' needs to approximate R_3 in order to make the divider ratio approximate $\frac{R_\mathrm{b}}{R_\mathrm{a}+R_\mathrm{b}}$.

---

3) Three-phase systems.

The 3-phase 4-wire supply system in this question gives phase-voltages that can be represented as phasors such that $u_\mathrm{a} = U\angle 0$, $u_\mathrm{b} = U\angle -2\pi/3$, and $u_\mathrm{c} = U\angle -4\pi/3$, where U is the rms (effective) value of the sinusoidal voltage.[5]

There are three *balanced* 3-phase loads: load 1 has impedances Z_1 in Δ (delta) connection; load 2 has impedances Z_2 in Y ('wye', or 'star') connection with the centre ('star-point') connected to the supply's neutral conductor; load 3 has impedances Z_3 3 star-connected with no neutral connection.

The only other two components shown are a single impedance Z_c from phase 'c' to neutral (the grounded conductor at the bottom), and another impedance Z_bc between phases b and c.

Reminder: the complex power into an impedance Z can be calculated from the applied voltage by

$$ui^* = u\left(\frac{u}{Z}\right)^* = \frac{uu^*}{Z^*} = \frac{|u|^2}{Z^*},$$

where u and i are the rms phasor voltage across and current through the impedance, using the passive convention. Note that the result is dependent only on the absolute value (not the phase) of the voltage.

a) Calculate complex powers into the loads.

Loads 1,2,3 (balanced loads).

Load 1: Δ-connected: Each impedance Z_1 gets voltage $\sqrt{3}U$, and therefore has complex power $S = \left(\sqrt{3}U\right)^2/Z_1^*$. There are three such loads, so the total complex power is $S_\mathrm{t} = 9U^2/Z_1^*$.

Load 2: Y-connected: Each impedance Z_2 gets voltage U, thus complex power $S = U^2/Z_2^*$. The total complex power is thus $S_\mathrm{t} = 3U^2/Z_2^*$.

Load 3: Y-connected but '4-wire' (with neutral). This is exactly the same principle as load 2, because the load is balanced, and therefore there is no current in the neutral conductor.[6] Thus the total complex power is $S_\mathrm{t} = 3U^2/Z_3^*$.

[5]Why have a phasor whose amplitude is an rms value instead of a peak value? This might sound wrong. We first introduced the idea of phasors by thinking of representing a signal like $A\cos(\omega t+\phi)$ as a complex quantity $A\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\phi}$ and then removing the term $\mathrm{e}^{\mathrm{j}\omega t}$ which is assumed to be the same for all currents and voltages in the circuit (in steady-state excitation by independent sources with frequency ω). In that case, A was the amplitude, but now we're talking about making the phasor's size represent the rms value – how can we do this? The reasoning is similar to why we can drop the $\mathrm{e}^{\mathrm{j}\omega t}$: if we divide all the currents and voltages in ac analysis by the *same* function, we don't change the solution. By removing the $\mathrm{e}^{\mathrm{j}\omega t}$ we got the advantage of simpler equations. By dividing all currents and voltages by $\sqrt{2}$ we still get the same equations to solve for the circuit, but we get the advantage that when we multiply a voltage and [conjugate]current to get power, the result doesn't have to be scaled by a factor of two: instead it immediately tells us the power. That is why people almost always use rms values for phasors, within the subject of electric power.

[6]If the loads were not balanced then the 3-wire case would be harder but the 4-wire case would be fairly easy as the voltages across all impedances would be known directly. In the unbalanced Y-connected 3-wire case, one does not immediately know the potential on the 'star-point' (centre-connection), and therefore has to use Y-Δ conversion or e.g. node-analysis to get a result.

We would have had nicer numbers if we'd used line voltage instead.[7] In general, for balanced three-phase load, if we know the line voltage U_L and the current I in each phase-conductor of the line, then the total apparent power is $\sqrt{3}U_\mathrm{L}I$.

Unbalanced loads Z_c and Z_bc:

Load c: $S = U^2/Z_\mathrm{c}^*$.

Load bc: $S = \left(\sqrt{3}U\right)^2/Z_\mathrm{bc}^*$.

b) The current i_c, expressed as a phasor.

If we write the full equations, in the way we would normally have done before we specifically studied 3-phase circuits, then we would sum together contributions due to 8 impedances that affect the current in the c-phase conductor. The only impedances that don't matter are the top Z_1 in the delta (because it doesn't couple to phase c) and the two top Z_2 in the 4-wire Y (they also are not affecting phase c). All the others would be included in the calculation, although we know that several of them will in fact cancel because of being balanced.

However, we now know clever tricks about 3-phase systems, useful in the *balanced* state. For the three balanced loads we can simply take the total complex power, then assume $\frac{1}{3}$ of this in each phase, and thereby calculate the phase currents by dividing this complex power by the phase voltage.

For the three balanced loads this gives a contribution to the phase-c current of

$$\left(\frac{\frac{1}{3}\left(\frac{9U^2}{Z_1^*}+\frac{3U^2}{Z_2^*}+\frac{3U^2}{Z_3^*}\right)}{U\underline{/-4\pi/3}}\right)^* = \left(\frac{3}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)\cdot U\underline{/-4\pi/3},$$

(the negative phase in the expression on the right comes from dividing by a negative phase [which gives positive phase] but then taking the complex conjugate [making it negative again!]...).

To this current we add the currents due to the two unbalanced loads, each of which is connected to phase c: these are

$$i_{\mathrm{Z_{bc}}} + i_{\mathrm{Z_c}} = \frac{u_\mathrm{c}-u_\mathrm{b}}{Z_\mathrm{bc}} + \frac{u_\mathrm{c}}{Z_\mathrm{c}} = \frac{1}{Z_\mathrm{bc}}\sqrt{3}U\underline{/\pi/2} + \frac{1}{Z_\mathrm{c}}U\underline{/-4\pi/3},$$

so the total is

$$i_\mathrm{c} = U\left[\left(\frac{3}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)1\underline{/-4\pi/3} + \frac{\sqrt{3}}{Z_\mathrm{bc}}1\underline{/\pi/2} + \frac{1}{Z_\mathrm{c}}1\underline{/-4\pi/3}\right].$$

c) The current i_n, as a phasor (i.e. caring about the size *and* phase). We could write this out in full detail, by adding the terms from the 4 impedances that connect to the neutral conductor (3 of Z_2 and 1 Z_c. But this isn't necessary: the load 2 is balanced, so its neutral current is zero as long as the voltage source is a balanced three-phase source (equal magnitudes, perfect 120° phase-shifts). Thus, the only neutral current is the current in Z_c. This is $u_\mathrm{c}/Z_\mathrm{c}$. As we don't know the angle of the impedance, it's fine to write this as just $\frac{1}{Z_\mathrm{c}}U\underline{/-4\pi/3}$. We could alternatively write $i_\mathrm{n} = \frac{U}{|Z_\mathrm{c}|}\underline{/-4\pi/3 - \underline{/Z_\mathrm{c}}}$. In reality we tend to care about the magnitude of a current or voltage, but not about its phase-angle unless we need to add different currents or voltages together (in which case the phases are obviously very important).

d) Given that the rms *phase-voltage* (from a phase to neutral [ground]) from the source is U, the absolute values of other voltage-definitions are:
i) peak phase-voltage: $\sqrt{2}U$ (because it's sinusoidal).
ii) rms line voltage (voltage between two phases): $\sqrt{3}U$.
iii) peak line voltage: $\sqrt{2}\sqrt{3}U$.

Note, again, that the *normal* power-system way of describing a three-phase system is to quote the line-voltage. Above the lowest voltage-levels one doesn't even use a neutral conductor, but has all loads connected between the phases (any neutral or 'ground' connection becomes important only in faults where there is a short-circuit). In this course we mainly use phase-voltage, because this gives us nicely defined potentials with respect to our 'reference node'.

[7] If U_L is the line-voltage (rms absolute value), then the power in a Δ-connected load of 3 identical impedances Z_Δ would be just $S_\mathrm{t} = 3U_\mathrm{L}^2/Z_\Delta^*$, and in the Y-connected load would be $S_\mathrm{t} = 3\left(U_\mathrm{L}/\sqrt{3}\right)^2/Z_\mathrm{Y}^* = U_\mathrm{L}^2/Z_\mathrm{Y}^*$.

4) More diodes. **Not a real "question".**

The only question in part 4) was the following:

Q: What would happen, even with the engine not running (no rotation) if someone accidentally connected a car's battery the wrong way round (+ and − poles reversed)?

A: It wouldn't be appreciated by the diodes. Nor by the battery. Current would flow from the battery, through the alternator's ground terminal (which would now be more positive than the other terminal), through three diodes in parallel then another three in parallel, and back to the battery. Of course, there may be bad effects on other equipment too, but we're here looking only at the alternator.

---

5) Coupled coils (mutual inductanec), transformers.

There wasn't a question 5 in the homework: not time for even more material!

But you're warmly recommended to look at the mutual inductance / transformer question from last year's homework.

Compiled: March 1, 2013, with small changes (missing j, changed explanations).
First version: 28 February, 2013.