KTH EI1120 Elkretsanalys (CENMI), Omtenta 2014-05-22 kl 08–13

Hjälpmedel: Ett A4-ark med studentens anteckningar (båda sidor).

Svar får anges på svenska eller engelska. En kort ordlista finns på sista sidan.

Läs varje tal noggrant **innan du försöker svara**.
Tänk på att **använda återstående tid till att kolla igenom varje svar**: man kan göra dimensionsanalys, rimlighetsbedömning (t.ex. ”är det rätt att y går ner medan x går ner?”), och lösning genom en alternativ metod. Lösningar ska **förenklas** om inte annat är specificerat.
Satsa inte för mycket tid på bara en uppgift om du fastnar: ta hänsyn till poängvärden på uppgifterna. Det är ofta så att **senare** *del*tal är betydligt **svårare** än de första deltalen.

Tentan har 8 tal i 3 delar: 3 i del A (12p), 2 i del B (10p) och 3 i del C (18p).
Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_k vara poängen man fick från kontrollskrivning KS1, och b_k poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a,a_\mathrm{k})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b,b_\mathrm{k})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a,a_\mathrm{k})+\max(b,b_\mathrm{k})+c+h}{A+B+C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan!
Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Är betyget mellan 44 och 50%, eller bara en av delarna av tentan underkänd trots bra betyg i de andra, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen. Se PM:et angående rättningsnormer och överklagande. Instruktionerna ovan tar prioritet över PM vid skillnad.

Examinator: Nathaniel Taylor

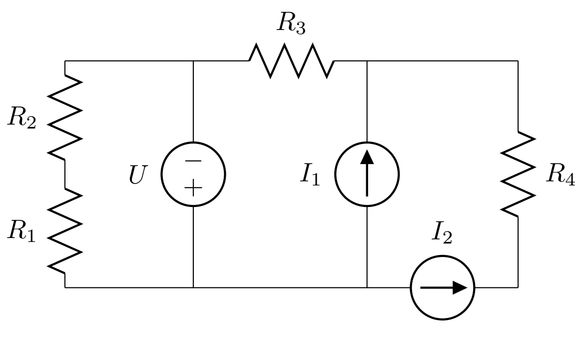
Del A. Likström

1) [4p]
Kända: R_1, R_2, R_3, R_4, U, I_1, I_2.

a) [1p] Bestäm effekten levererat till R_1.

b) [1p] Bestäm effekten levererat till R_3.

c) [2p] Bestäm effekten levererat från källan U.

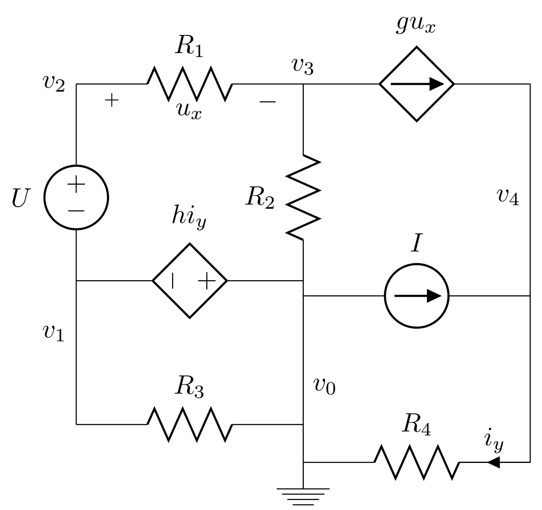


2) [4p]
Kända: R_1, R_2, R_3, R_4, U, I, g, h.

Använd nodanalys för att skriva ekvationer som går att lösa för de okända potentialerna v_1, v_2, v_3, v_4.

Du *måste inte lösa* ekvationerna, och *måste inte* skriva om dem i förenklad eller matris form. Du får definiera hjälpvariabler (men ekvationerna måste räcka till att unikt bestämma potentialerna).

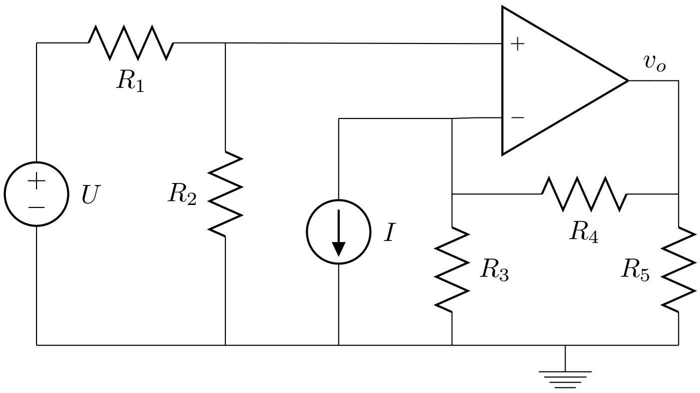
Det finns flera möjliga svar (alla med samma lösning). Förmodligen är det bäst att använda ett systematiskt sätt att skriva ekvationerna ...



3) [4p]
Kända: U, I, R_1, R_2, R_3, R_4, R_5.
Operationsförstärkaren antas vara idéal.

a) [3,5p] Bestäm v_o.

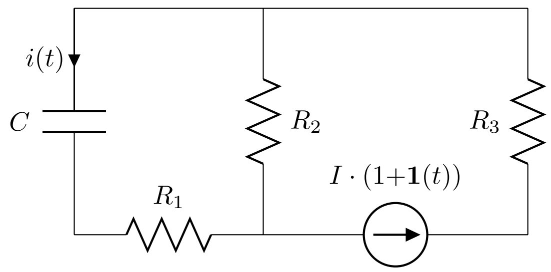
b) [0,5p] Vilken komponent är inte relevant till lösningen av del 'a)'? Varför?



Del B. Transient

4) [5p]
Kända: I, C, R_1, R_2, R_3.
Kretsen är i jämviktsläge innan tiden $t = 0$.
Enhetsstegfunktionen är skriven här som $\mathbf{1}(t)$.

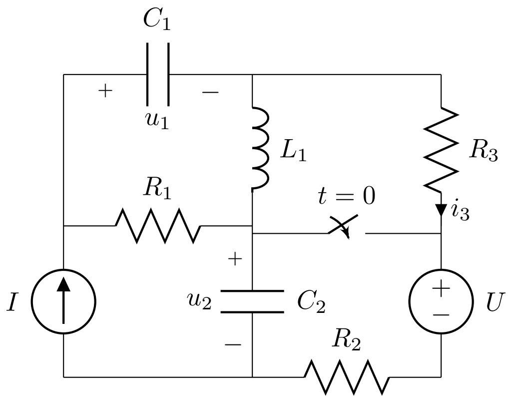
Bestäm strömmen $i(t)$ i kondensatorn, som tidsfunktion för perioden $t > 0$.



5) [5p]
Kända: I, U, R_1, R_2, R_3, L_1, C_1, C_2.

a) [3p] Betrakta jämviktsläget vid $t = 0^-$.
Bestäm $i_3(0^-)$, $u_1(0^-)$, och $u_2(0^-)$.

b) [2p] Betrakta tiden $t = 0^+$.
Bestäm $u_2(0^+)$ och $i_3(0^+)$.

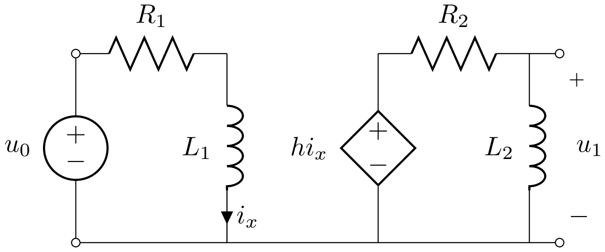


Del C. Växelström

6) [6p]
Kända: R_1, L_1, h, R_2, L_2.
Spänningen u_0 på polerna till vänster orsakar spänningen u_1 mellan polerna till höger, vilka är öppna (ingen ström).

a) [3p] Bestäm kretsens nätverksfunktion,

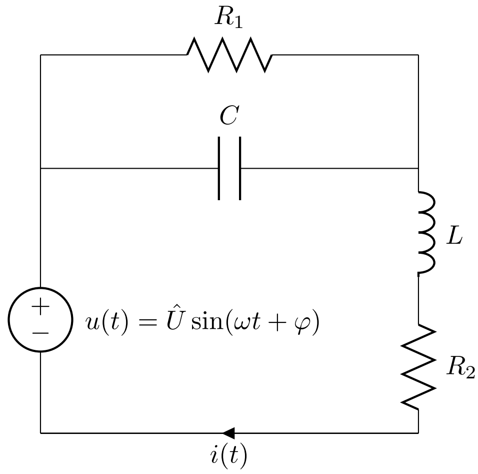
$$H(\omega) = \frac{u_1(\omega)}{u_0(\omega)}.$$



b) [3p] (Observera att funktionen $H'(\omega)$ i detta deltal **inte** är lika med funktionen $H(\omega)$ från deltal 'a)'!)
Skissa ett Bode amplituddiagram, på antagandet $\omega_1 \ll \omega_2 \ll \omega_3 \ll \omega_4$, och $k < 1$, av funktionen

$$H'(\omega) = k\frac{(1 + \mathrm{j}\omega/\omega_1)(1 + \mathrm{j}\omega/\omega_4)}{(1 + \mathrm{j}\omega/\omega_2)(1 + \mathrm{j}\omega/\omega_3)}.$$

7) [6p]
Kända: $\hat{U}$, ω, φ, R_1, C.
Observera att beräkningarna här kan göras med växelströmsanalys (komplexa tal).



a) [3p] Bestäm R_2 och L för att maximera effektutvecklingen i R_2.

b) [3p] Bestäm den genomsnittliga effekten (”aktiveffekt” i växelströms terminologi) i R_2. Anta att R_2 och L är bestämd enligt deltal ’a)’, och att man kan därför uttrycka lösningen med bara de kända variablerna.
Har du inte gjort deltal ’a)’, så kan du skriva uttrycket med R_2 och L också (det blir då lite avdrag för mindre förenklad svar).

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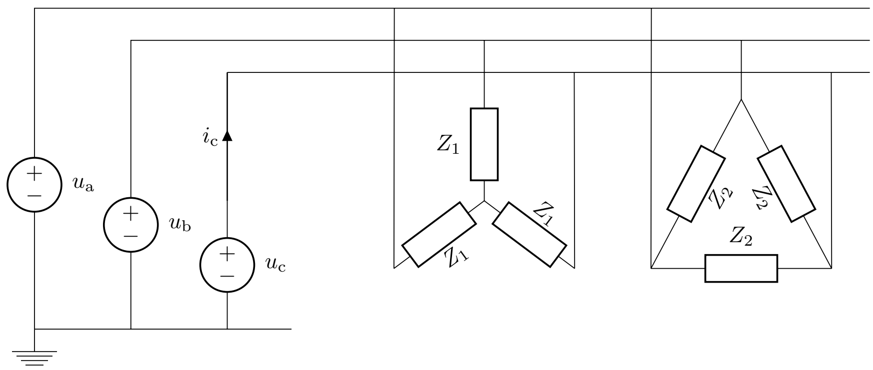
8) [6p]
Kända: U, ω, R, L, C.
Källorna i diagrammet är växelströmskällor med vinkelfrekvens ω och spänning (effektivvärde) U.
De kan beskrivas med fasvektorer $u_\text{a} = U\angle 0$, $u_\text{b} = U\angle{-120°}$ och $u_\text{c} = U\angle{-240°}$.
Varje impedans Z_1 är en spole L, och varje impedans Z_2 är en seriekopplade mostånd R och kondensator C.



a) [3p] Vilken aktiveffekt försörjs av hela trefaskällan (alla tre spänningskällor)?

b) [2p] Vilket värde måste L ha (uttryckt i ω, R, och C) för att ingen reaktiveffekt dras från källan?

c) [1p] Bestäm i_c (magnitud och fas) när L är valt enligt deltal ’b’.

Ordlista över mindre självklara översättningar: current *ström*, voltage *spänning*, power *effekt*, rms value *effektivvärde*, phasor *fasvektor*, source *källa*, unit-step *enhetssteg*, terminal *pol*, opamp (operational amplifier) *operationsförstärkare*, angular(radian) frequency *vinkelfrekvens*, equilibrium *jämviktsläge*, inductor (coil) *spole*, active power *aktiveffekt.*

Solutions (EI1120, VT14, 2014-05-22)

Q1

It is useful to remember that the power dissipation in a resistor R is i^2R or u^2/R, regardless of the direction (sign) of current or voltage.

a) The series combination of R_1 and R_2 is connected directly across a voltage source, U, so the remainder of the circuit is irrelevant.

The magnitude of current in R_1 is $\frac{U}{R_1+R_2}$, so the power is $\left(\frac{U}{R_1+R_2}\right)^2 R_1$, giving $P_{R_1} = \frac{U^2R_1}{(R_1+R_2)^2}$.

b) Consider KCL in the right-hand node of R_3: the current from right to left in R_3 must be $I_1 + I_2$. Thus, $P_{R_3} = (I_1+I_2)^2 R_3$.

c) There are two paths that current from source U can travel: one is through the resistors R_1 and R_2, and the other is through R_3 and the current-sources; these currents have been used already in the above two questions.

The current out from the + terminal of U is therefore seen to be $I_U = \frac{U}{R_1+R_2} + I_1 + I_2$, so the power delivered by the voltage source is $P_U = \frac{U^2}{R_1+R_2} + UI_1 + UI_2$.

Q2

Let's just show the "simple" method (of writing lots of equations, systematically, without any simplifications).

First, KCL at all nodes except the ground node. The currents in the voltage sources are not known, so we define new unknowns: the current into the + of the independent source U can be called i_u, and the current into the + of the dependent source hi_y can be called i_h.

$$\begin{aligned}
\frac{v_1}{R_3} - i_u - i_h &= 0 \qquad \text{KCL}(1)_{\text{out}} \\
\frac{v_2 - v_3}{R_1} + i_u &= 0 \qquad \text{KCL}(2)_{\text{out}} \\
\frac{v_3 - v_2}{R_1} + \frac{v_3}{R_2} + gu_x &= 0 \qquad \text{KCL}(3)_{\text{out}} \\
\frac{v_4}{R_4} - gu_x - I &= 0 \qquad \text{KCL}(4)_{\text{out}}
\end{aligned}$$

Each voltage source has introduced a new unknown, hence the i_u and i_h in the above. But each voltage source also gives a further equation, relating two node-potentials. Write these equations:

$$\begin{aligned}
v_2 - v_1 &= U \\
v_1 &= -hi_y.
\end{aligned}$$

The dependent sources depend on further variables that are not known, u_x and i_y. Use circuit properties to define these controlling variables in terms of components (known) and node potentials (unknowns that we already have in our equations),

$$\begin{aligned}
u_x &= v_2 - v_3 \\
i_y &= \frac{v_4}{R_4}.
\end{aligned}$$

Now the unknowns are v_1, v_2, v_3, v_4, u_x, i_y, i_u and i_h (total 8). Lo and behold — there are 8 equations, too. And if the circuit is a sensible one, we can expect these equations to be linearly independent, giving a solvable system. If we had tried more inventive methods of setting up the equations, we would have to think hard about ensuring the independence.

Q3

a) By the usual assumption of an ideal opamp with negative feedback, $v_- = v_+$.

By voltage division, $v_- = v_+ = \frac{UR_2}{R_1+R_2}$.

Node analysis with $\text{KCL}_{(\text{out})}$ at the inverting input gives $\frac{v_-}{R_3} + I + \frac{v_- - v_o}{R_4} = 0$, so $v_o = v_- \frac{R_3+R_4}{R_3} + IR_4$.

Putting these together, $v_o = U\frac{R_2(R_3+R_4)}{R_3(R_1+R_2)} + IR_4$.

b) The resistor R_5 does not appear in the solution above. It can be seen to be irrelevant as it is connected in parallel with a voltage source. The voltage source is the one "inside" the opamp (in other words, "modelling the opamp"), which also connects between opamp output and ground.

Another way of thinking is that the current in R_5 has no effect on the opamp inputs, so the opamp output will be whatever is needed to force the inputs to the same potential; there is no feedback path through R_5, so this component is irrelevant to the question of what the output voltage is (it *is* relevant to another question: what is the output *current* ... but that is not the question here).

Q4

As there is just one reactive component, the rest of the circuit can be reduced to an equivalent source: let's use a Thevenin equivalent. Remove the capacitor, and find the Thevenin equivalent at the terminals where it connected. The open-circuit voltage is $U_\mathrm{T} = I \cdot (1 + \mathbf{1}(t))\, R_2$; this can be found by considering that if the capacitor is replaced with an open circuit, then all the current from the source must go through R_2, and R_1 has zero voltage. The resistance is $R_\mathrm{T} = R_1 + R_2$, which can be seen by looking at the circuit impedance between the terminals when the source is "set to zero".

In the equilibrium at $t = 0^-$, the capacitor voltage will equal the source voltage before the step, i.e. it will be IR_2. The final equilibrium as $t \to \infty$ will give a capacitor voltage of equal to the source's open-circuit voltage, i.e. $v_\mathrm{c}(\infty) = U_\mathrm{T}(\infty) = 2IR_2$. The time-constant is $CR_\mathrm{T} = C\,(R_1 + R_2)$. The final current must be zero, $i(\infty) = 0$, as the source is charging the capacitor. The initial current is $i(0+) = \frac{2IR_2 - IR_2}{R_1+R_2} = \frac{R_2}{R_1+R_2} I$, driven by the difference between the source and capacitor voltages, acting on the source resistance. So the time-function after the step is $i(t) = \frac{R_2}{R_1+R_2}\, I \,\exp \frac{-t}{C(R_1+R_2)}$.

So, $i(t) = \frac{R_2}{R_1+R_2}\, I \,\exp\left(\frac{-t}{C(R_1+R_2)}\right)$, (for $t > 0$).

Q5

a)

When the switch is open and capacitors are in equilibrium, there is just one path around the circuit, which includes the current source and the marked current i_3; checking the direction, we see that $i_3(0^-) = I$.

KVL in the loop with R_1 and L_1 shows that $u_1(0^-) = IR_1$, as there is zero voltage across the inductor in equilibrium.

By KVL in a wisely chosen loop, it is seen that $u_2(0^-) = U + I(R_2 + R_3)$. Note that the current-source's voltage is not initially known (e.g. it is not guaranteed to be zero), so we need to do KVL around L_1, R_3, U, R_2.

b)

A capacitor's voltage has continuity: therefore, from part 'a)', $u_2(0^+) = U + I(R_2 + R_3)$.

The final part, finding $i_3(0^+)$, is more difficult. First replace all inductors and capacitors with (respectively) current and voltage sources having the values calculated at $t = 0^-$ (continuity).

Then it is convenient to choose the node below R_3 as the reference (ground). The source I must draw all its current through R_2 and C_2; these two components connect back to the ground node without any other connection to parts of the circuit that affect i_3. So the three components C_2, R_2 and U can be ignored, and we assume the input terminal of source I is connected directly to the ground node.

The remaining circuit is just the ground node and a supernode. The supernode is the nodes on the two sides of C_1, which for this analysis at $t = 0^+$ can be replaced with a voltage source of value $u_1 = IR_1$, as found in part 'a)'. Defining the potential of the node above R_3 to be v, $\mathrm{KCL}_{(\mathrm{out})}$ in the supernode gives $\frac{v}{R_3} - I - I + \frac{v + IR_1}{R_1} = 0$, as the inductor is replaced by a current source I which is the current in the inductor in the equilibrium $t = 0^-$.

Hence, $v\left(\frac{1}{R_1} + \frac{1}{R_3}\right) = I$, and $i_3 = \frac{v}{R_3}$; combining these gives $i_3(0^+) = \frac{IR_1}{R_1+R_3}$.

Q6

a) The three components on the left are independent of the three on the right: there is only one node in common between the two sides, and the dependent source is controlled *by* a variable on the left side.

The left side can be analysed by itself to find the controlling variable i_x, which then can be inserted to solve the right side. (Or just write the node equations and see the same result.)

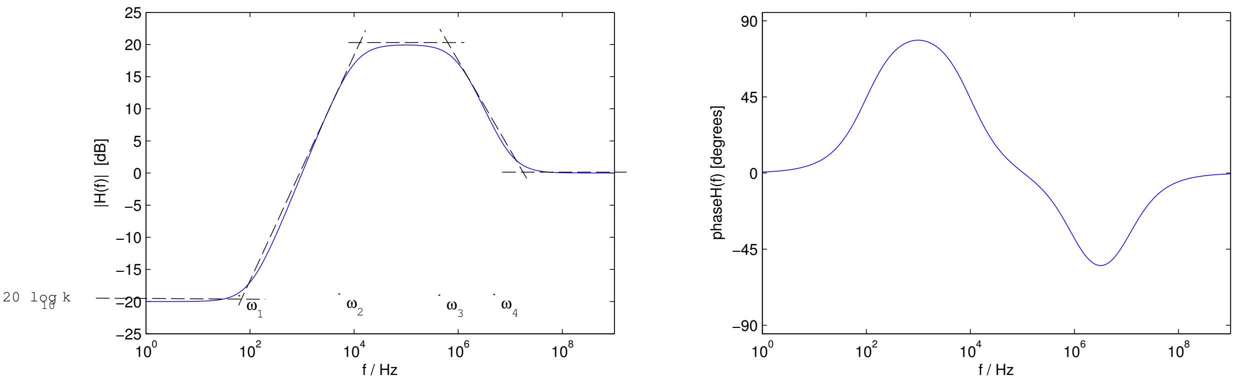
The left: $i_x = \frac{u_0}{R_1+\mathrm{j}\omega L_1}$.

The right: voltage division: $u_1 = hi_x\frac{\mathrm{j}\omega L_2}{R_2+\mathrm{j}\omega L_2}$.

Combined, $\frac{u_1}{u_0} = h\frac{\mathrm{j}\omega L_2}{(R_1+\mathrm{j}\omega L_1)(R_2+\mathrm{j}\omega L_2)}$.

b) Bode amplitude plot of the function $k\frac{(1+\mathrm{j}\omega/\omega_1)(1+\mathrm{j}\omega/\omega_4)}{(1+\mathrm{j}\omega/\omega_2)(1+\mathrm{j}\omega/\omega_3)}$. This has two poles and two zeros.

The following figure is only an example! The numbers are just ones that were chosen as one example of parameters that fit the given conditions; for example, I've chosen $k = 0.1$, $\omega_1 = 2\pi \cdot 100\,\mathrm{Hz}$, etc. The figure also shows a phase-plot on the right: this was not required, but is included for interest ... one day we might have a phase-plot question in an exam. The poor quality of the figures is due to use of Matlab (and my not having all day to fool around trying to make its output look like what's on the screen); one day I must learn a decent program for plotting, such as SciPy Matplotlib or such.



The given relation of $\omega_1 \ll \omega_2 \ll \omega_3 \ll \omega_4$ determines that, from left to right, there is a flat part, then up, then another flat part, then down, then another flat part. The vertical position of the third flat part could be anywhere below the position of the second one, depending on the actual values of the frequency-constants. If the order of the frequency-constants were different, e.g. $\omega_2 \ll \omega_1$, then the shape would be qualitatively different. If the frequency-constants were closer together (just '$<$', not '$\ll$') then we would not even get a straight line in the plot between the different frequencies.

Explanation of the shape. For $\omega \ll \omega_1$, all four pole or zero terms are approximately 1, so the gain is just k. This should be marked on the y-axis as a dB value of $20\log_{10} k$. Increasing the frequency, there is first a zero, so the amplitude plot starts increasing at 20 dB/decade; then there is a pole, which cancels the effect of the zero, giving a flat response again. The next change is another pole, giving a response of -20 dB/decade, i.e. falling, until the final zero is reached, after which the zeros and poles cancel (2 zeros and 2 poles), giving a flat response again. The value of the function for $\omega \gg \omega_4$ will be $\frac{k\omega_2\omega_3}{\omega_1\omega_4}$, but there won't be removal of points for omitting this on the sketch.

Q7

a) We are told that R_1 and C (and the source) are known, but that we can choose R_2 and L, with the aim of maximising the power into R_2 (the power into R_2 is of course purely real, as it is a pure resistor).

This is therefore a classic maximum power question, with source-impedance of $Z_1 = \frac{1}{\left(\frac{1}{R_1}+\mathrm{j}\omega C\right)}$.

The ac maximum power principle is that the load, $Z_2 = R_2 + \mathrm{j}\omega L$, must be the complex conjugate of the source impedance in order to give maximum active power transfer to the load.

Therefore, we want to choose R_2 and L so that $Z_1 = Z_2^*$, or equivalently, $Z_1^* = Z_2$.
Being wise, we look ahead: Z_2 already has these variables R_2 and L separated between its real and imaginary parts; we therefore keep Z_2 as it is, and manipulate Z_1 so that was can separate its real and imaginary parts and equate them with the corresponding parts of Z_2.

$$Z_2 = R_2 + \mathrm{j}\omega L = Z_1^* = \left(\frac{1}{\frac{1}{R_1} + \mathrm{j}\omega C}\right)^* = \frac{1}{\frac{1}{R_1} - \mathrm{j}\omega C} = \frac{R_1 + \mathrm{j}\omega C R_1^2}{1 + \omega^2 C^2 R_1^2}.$$

Equating the real and imaginary parts, $R_2 = \frac{R_1}{1+\omega^2C^2R_1^2}$ and $L = \frac{CR_1^2}{1+\omega^2C^2R_1^2}$.

b) The total circuit impedance seen by the source is $Z_1 + Z_2$. When the values of R_2 and L are chosen according to part 'a)', above, then we know $Z_1^* = Z_2$, so the total circuit impedance is $Z_1 + Z_1^*$. This simplifies to $2\Re\{Z_1\}$, where $\Re\{\}$ indicates the real part.

To find the power dissipated in R_2, it is sufficient to know the rms current. The load components (R_2 and L) are in series, so the current in R_2 is the same as the marked current i. There is no need to care about the phase angle when calculating power due to a current in a resistor: only the magnitude is important. The magnitude of the current around the circuit is

$$i = \frac{\hat{U}}{2\Re\{Z_1\}} = \frac{\hat{U}}{2R_2} = \hat{U}\frac{1+\omega^2C^2R_1^2}{2R_1},$$

which is a peak value, because $\hat{U}$ is a peak value.

The mean power dissipation (active power) in the load is $\frac{1}{2}|i|^2R_2$.
The factor $1/2$ is needed because the i that we calculated above is a peak value.

Putting these together, $P = \frac{\hat{U}^2}{2} \cdot \frac{\left(1+\omega^2C^2R_1^2\right)^2}{4R_1^2} \cdot \frac{R_1}{1+\omega^2C^2R_1^2}$.

The solution is then $P = \frac{\hat{U}^2\left(1+\omega^2C^2R_1^2\right)}{8R_1}$.

Q8

a) Each impedance Z_1 is $\mathrm{j}\omega L$, and has rms voltage U across it, and there are three such impedances.

By the relation $S = \frac{|u|^2}{Z^*}$, the total complex power to the three Z_1 is $S_1 = \frac{3U^2}{-\mathrm{j}\omega L} = \mathrm{j}\,\frac{3U^2}{\omega L}$.
We note that this is purely imaginary, so there is no active power, which means we didn't really need to calculate it — we could just have said "there will be no active power in an [ideal] inductor".

Each impedance Z_2 is $R + \frac{1}{\mathrm{j}\omega C}$, and the voltage across it is $\sqrt{3}U$ due to the delta connection.

The complex power into the three impedances Z_2 is therefore $S_2 = 3\frac{\left(\sqrt{3}U\right)^2}{R-\frac{1}{\mathrm{j}\omega C}} = \frac{9U^2\left(R-\mathrm{j}\frac{1}{\omega C}\right)}{R^2+(\omega C)^{-2}}$.

The total complex power is the sum of the above two contributions, $S_1 + S_2$, and the total active power is the real part of this. Because the Z_1 gives no active power, we just need to consider the real part of S_2.
That gives $P = \frac{9U^2R}{R^2+(\omega C)^{-2}}$.

b) Choose L to set the reactive power from the source to be zero. This means that we want $\Im\{S_1 + S_2\} = 0$, where $\Im\{\}$ mean the imaginary part.

From the above expressions for reactive power (or by considering that Z_1 is inductive, and Z_2 is partially capacitive) we see that the two loads have opposite sign of reactive power. The task is therefore to set their reactive powers to have equal magnitude, so that they cancel.

The required equality is $\frac{3U^2}{\omega L} = \frac{9U^2\frac{1}{\omega C}}{R^2+(\omega C)^{-2}}$, so the required inductance is $L = \frac{R^2+(\omega C)^{-2}}{3}C$.

c) It is known (from part 'b)') that the load is balanced and the reactive power from the source is zero. The reactive power from each individual voltage source must therefore be zero, so the current i_c must have the same phase-angle as the voltage u_c. (Note that in general, a zero reactive power could also be acheived by a phase-shift of π, or by the current having zero magnitude: but we exclude these situations here because we know that Z_2 is partly resistive and is therefore consuming active power.)

This fixes the phase-angle to $\angle i_c = \angle u_c = -240^\circ = 120^\circ$.

The magnitude of i_c is whatever is needed to provide one third of the total active power consumption of the loads, i.e. $P/3$ where P is from part 'a)'. This is $|i_c| = \frac{9U^2R}{R^2+(\omega C)^{-2}} / (3U) = \frac{3UR}{R^2+(\omega C)^{-2}}$.

The complex current in phase c is therefore $i_c = \frac{3UR}{R^2+(\omega C)^{-2}}\angle{-240^\circ}$.