

EI1110 Elkretsanalys, Kontrollskrivning KS1, 2014-09-30 kl 15–17

Hjälpmedel: Ett A4-ark med godtyckligt innehåll (handskriven, datorutskrift, diagram, m.m.).

Kontrollskrivningen har 3 tal, med totalt 12 poäng. Den omfattar ämnet 'Likström' och motsvarar del A i tentamen. Det högre av betygen från KS1 (den här) och från tentans del A kommer att användas vid betygsättning av tentan. (Del A är godkänt vid $\geq 40\%$, men glöm inte att tentan kräver också minst 50% räknat över talen i alla delar.)

Om inte annan information anges i ett tal, ska: angivna värden av komponenter (t.ex. R för en motstånd, U för en spänningskälla) antas vara kända storheter; och andra storheter (t.ex. strömmen markerad i en motstånd) antas vara okända storheter; och komponenter antas vara idéala.

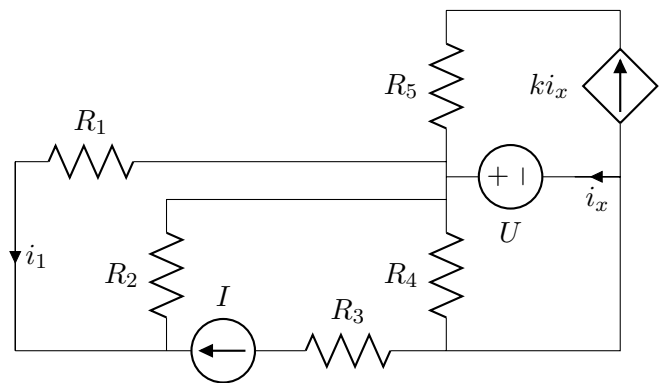
Lösningar ska uttryckas i kända kvantiteter, och förenklas. Var tydlig med diagram och definitioner av variabler. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Använd återstående tid för att kolla på svaren!

Examinator: Nathaniel Taylor

1) [4p]

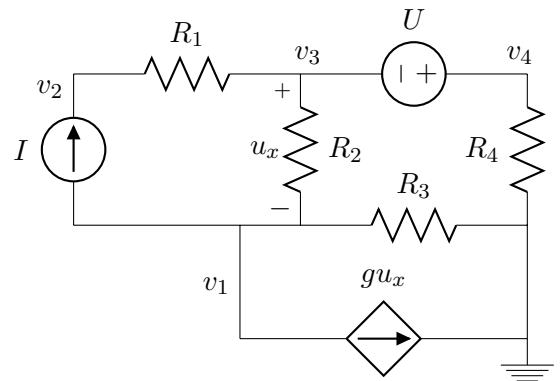
- a) [1p] Vilken effekt absorberas av R_3 ?
- b) [1p] Vad är den markerade strömmen i_1 ?
- c) [1p] Vilken effekt absorberas av R_4 ?
- d) [1p] Vilken effekt levereras av källan U ?



2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1 , v_2 , v_3 och v_4 .

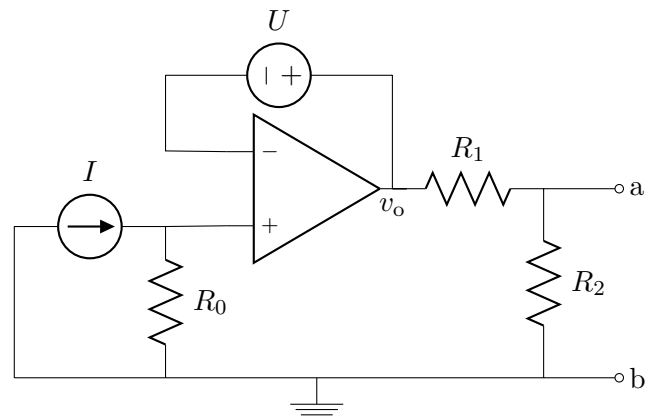
Du *måste inte* lösa eller förenkla ekvationerna: du behöver bara visa att du kan översätta från kretsen till ekvationerna.



3) [4p]

- a) [2p] Vad är potentialen på pol 'a'? (D.v.s. spänningen mellan 'a' och 'b').

- b) [2p] Vad är den störste effekten som kan levereras från denna krets till en valfri extern krets som kopplas mellan polerna 'a'-'b'? (Du får tänka på den externa kretsen som ett enda motstånd om det hjälper: slutsatsen är samma för motstånd eller mer allmänt fall.)

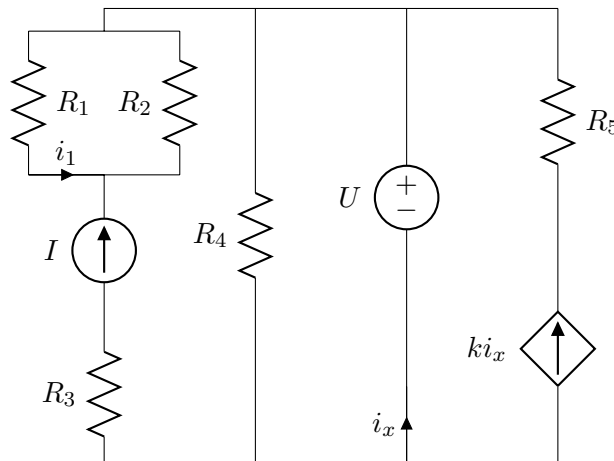


Solutions, EI1110 KS1 2014-09-30

1) a) $P_{R_3} = I^2 R_3$, b) $i_1 = \frac{-IR_2}{R_1 + R_2}$, c) $P_{R_4} = U^2/R_4$, d) $P_U = U i_x = U \frac{-I + U/R_4}{1 + k}$.

The following re-drawing may help in considering the above results.

The final one was obtained by simplification and nodal analysis, taking KCL just at the voltage source + pole, into three simplified parallel branches.



2)

Extended nodal analysis (“the simple way”)

Let’s define the unknown current in voltage source U as i_α , going into the source’s + terminal.

KCL (outgoing currents) at all nodes except ground:

$$\text{KCL(1)} : 0 = I + g u_x + \frac{v_1 - v_3}{R_2} + \frac{v_1}{R_3} \quad (1)$$

$$\text{KCL(2)} : 0 = -I + \frac{v_2 - v_3}{R_1} \quad (2)$$

$$\text{KCL(3)} : 0 = \frac{v_3 - v_2}{R_1} + \frac{v_3 - v_1}{R_2} - i_\alpha \quad (3)$$

$$\text{KCL(4)} : 0 = i_\alpha + \frac{v_4}{R_4} \quad (4)$$

Now there are 6 unknowns ($v_1, v_2, v_3, v_4, u_x, i_\alpha$), and 4 equations.

Next, include the further information given by the voltage source,

$$v_4 - v_3 = U, \quad (5)$$

and also define the controlling variable u_x in terms of the existing known or unknown quantities,

$$u_x = v_3 - v_1 \quad (6)$$

The systematic way in which this was done is important! There *are* plenty of ways to write a sufficient set of equations, but it is also dangerously easy to write some linearly dependent equations and assume that “ n unknowns, n equations, therefore it’s all ok” ... the above method is useful!

Alternative: Simplifications and/or Supernode

It's quite a nice circuit to solve if we had to do the solution by hand. The supernode approach would be good. (However, the method used above is better when you *don't* have to solve the equations, such as in this type of exam question or for solutions by computer-algebra system: that's because it minimises how much manipulation you do of the equations.)

The “standard procedure” in the supernode method is to notice that $v_4 = v_3 + U$, and therefore to include only one of these potentials in the equations that we have to solve; the other can easily be found after the solution. Then KCL is taken over both of these nodes together, to avoid dealing with the unknown current in the voltage source U . Let's choose to use v_3 ; then if the potential v_4 is needed in our equations (e.g. for KCL with R_4) we can write it as $v_3 + U$. (The reason for choosing to use v_3 instead of v_4 was that v_3 will obviously appear more in the equations, since there are several resistors connected to node 3, and a controlling variable that depends on this potential.)

The controlled source gu_x can be written as $g(v_3 - v_1)$ so that the variable u_x isn't used.

The branch of I and R_2 behaves, of course, like a current source I , because the series resistor is irrelevant when seen from outside this branch. The solution of the rest of the circuit only needs to know that I comes out of node 1 and into node 3. After the other potentials have been solved, we can find v_2 by $v_2 = v_3 + IR_1$: the resistor in series with a current source is, in effect, a known voltage source, because the voltage across it is completely determined by the current and resistance! This special treatment of the left branch is not a standard procedure in the supernode-method: it's just something that we can notice as a simplification, with the advantage of reducing the number of equations that have to be handled at the same time.

Node 1 is a further node, with unknown potential v_1 .

There are now only two unknown variables: v_1 and v_3 .

Writing KCL (out) at the supernode of nodes 3 and 4, and at node 1,

$$0 = -I + \frac{v_3 - v_1}{R_2} + \frac{v_3 + U}{R_4} \quad (7)$$

$$0 = I + g(v_3 - v_1) + \frac{v_1 - v_3}{R_2} + \frac{v_1}{R_3} \quad (8)$$

To do as the question required, one should also write as equations the earlier statements that would let us define v_2 and v_4 after the above equations are solved for v_1 and v_3 ,

$$v_4 = v_3 + U \quad (9)$$

$$v_2 = v_3 + IR_1 \quad (10)$$

Computer-assisted check of solutions to Question 2

We can choose some arbitrary numeric values:

$U = 12\text{ V}$, $I = 1.7\text{ A}$, $g = 0.9\text{ S}$, $R_1 = 8\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 33\ \Omega$, $R_4 = 4.7\ \Omega$,
and then compare solutions of our equations and a circuit-solver program.

The circuit can be described by the following “netlist” for solving in SPICE.

```
EI1110_HT14_KS1Q2
V1 4 3 DC 12.0
I1 1 2 DC 1.7
G1 1 0 3 1 0.9
R1 2 3 8.0
R2 3 1 15.0
R3 0 1 33.0
R4 4 0 4.7
.OP
.PRINT DC V(0) V(1) V(2) V(3) V(4)
.END
```

The result from putting the above into SPICE 2g.6 [1983-03-15!] is

```
node potentials:
( 1)  -1.3975   ( 2)  10.2133   ( 3)  -3.3867   ( 4)   8.6133

current in voltage source V1      -1.833E+00
current from controlled i-source  -1.79E+00
total power dissipation           4.17E+01
```

Putting our equation systems directly into Matlab symbolic toolbox, the 'extended method' gives

```
%% solve the equation-system symbolically, for the 6 listed unknowns
% (put an underscore onto variable "I", to avoid it being treated as an imaginary unit
% in matlab symbolic toolbox)
s = solve( ...
'0 = I_ + g*ux + (v1-v3)/R2 + v1/R3', ...
'0 = -I_ + (v2-v3)/R1', ...
'0 = (v3-v2)/R1 + (v3-v1)/R2 - ialph', ...
'0 = ialph + v4/R4', ...
'v4 - v3 = U', ...
'ux = v3 - v1', ...
'v1, v2, v3, v4, ux, ialph' )
s = simplify(s);

%% now show all the results of the 6 unknowns
for f=fields(s)', disp(f{1}), s.(f{1}), end
% ialph: -(U + I_*R2 - I_*R2*R3*g)/(R2 + R3 + R4 - R2*R3*g)
% ux: (I_*R2*R3 - R2*U + I_*R2*R4)/(R2 + R3 + R4 - R2*R3*g)
% v1: -(R3*U + I_*R2*R3 - R2*R3*U*g + I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v2: (I_*R1*R2 - R3*U - R2*U + I_*R1*R3 + I_*R1*R4 + I_*R2*R4 + R2*R3*U*g -
%      I_*R1*R2*R3*g - I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v3: -(R2*U + R3*U - I_*R2*R4 - R2*R3*U*g + I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v4: (R4*U + I_*R2*R4 - I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)

%% set numeric values for comparison with the SPICE solution
U = 12, I_ = 1.7, g = 0.9, R1 = 8, R2 = 15, R3 = 33, R4 = 4.7
% find the result of substituting the above values into the symbolic expressions
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))))); end
% ialph: -1.832612
% ux: -1.989180
% v1: -1.397543
% v2: 10.213276
% v3: -3.386724
% v4: 8.613276
```

A similar treatment of the equations derived from the supernode and simplification methods also matches the above.

```
%% solve for potentials
ss = solve( ...
'0 = -I_ + (v3-v1)/R2 + (v3+U)/R4', ...
'0 = I_ + g*(v3-v1) + (v1-v3)/R2 + v1/R3', ...
'v4 = v3 + U', ...
'v2 = v3 + I_ * R1', ...
'v1, v2, v3, v4' )
ss = simplify(ss)

ss.v1, ss.v2, ss.v3, ss.v4
% v1: -(R3*U + I_*R2*R3 - R2*R3*U*g + I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v2: (I_*R1*R2 - R3*U - R2*U + I_*R1*R3 + I_*R1*R4 + I_*R2*R4 + R2*R3*U*g -
%      I_*R1*R2*R3*g - I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v3: -(R2*U + R3*U - I_*R2*R4 - R2*R3*U*g + I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)
% v4: (R4*U + I_*R2*R4 - I_*R2*R3*R4*g)/(R2 + R3 + R4 - R2*R3*g)

U = 12, I_ = 1.7, g = 0.9, R1 = 8, R2 = 15, R3 = 33, R4 = 4.7
for f=fields(ss)', fprintf(' %s: %f\n', f{1}, double(subs(ss.(f{1}))))); end
```

% v1: -1.397543
% v2: 10.213276
% v3: -3.386724
% v4: 8.613276

3)

a)
$$u_{ab} = (IR_0 + U) \frac{R_2}{R_1 + R_2}$$

Probably the easiest way is to find v_o by putting together the equations $v_+ = IR_0$, $v_- = v_o - U$ and $v_+ = v_-$; then voltage division between R_1 and R_2 can be used to find the voltage across R_2 (the same as the voltage between 'a' and 'b').

We can treat v_o as a voltage source, independent of the values of R_1 and R_2 or of what is connected between 'a' and 'b'. As long as there is a negative-feedback path, the opamp will ensure that its output is whatever value is needed to ensure $v_- = v_+$; this will not be affected by how much current is taken from the output.

Another way is, of course, nodal analysis. It's probably easiest to take a supernode-based approach if we have to solve the equations by hand. The inverting input would, in this circuit, be able to be treated as part of the ground supernode: that's because it's connected through a voltage source to the opamp output, which is itself regarded as being connected to ground through its internal "controlled voltage source" model. There are then only two nodes where KCL is needed, and these are both trivial nodes (only two connections). After substituting $v_+ = v_- = v_o - U$, we get $-I - \frac{v_o}{R_0} = 0$, and $\frac{v_a - v_o}{R_1} + \frac{v_a}{R_2} = 0$.

b)
$$R_x = \frac{R_1 R_2}{R_1 + R_2}$$

Maximum power transfer from a two-terminal circuit [with finite, positive Thevenin resistance] is when the current is half of the short-circuit current, which is also when the voltage is half of the open-circuit voltage. A less general way of seeing this is that maximum power transfer occurs when a resistance equal to the circuit's Thevenin resistance is connected to the circuit. We must therefore find our circuit's Thevenin resistance: the desired R_x is this value.

It has already been described that v_o behaves as an ideal voltage source, not affected by the current into R_1 . The Thevenin resistance of this circuit seen from terminals 'a' and 'b' is therefore the same as if the opamp were removed and an independent voltage source v_o were inserted between ground and the left of R_1 . In this case the Thevenin resistance could be found by setting the source to zero and looking at the circuit's resistance between the two terminals, which would be the parallel sum of R_1 and R_2 .

If the reasoning in the above method is unclear, the more general method can be used: e.g. try nodal analysis on the circuit again, but now with the terminals 'a' and 'b' *short-circuited*, to find the short-circuit current. Then find the Thevenin resistance as the ratio of open-circuit voltage to short-circuit current: the open-circuit voltage was already calculated in part 'a').

c)
$$P_{\max} = \frac{R_2 (U + IR_0)^2}{4R_1 (R_1 + R_2)}$$

This can be derived from the condition of $i_{ab, \max \text{power}} = i_{ab, sc}/2$ or $u_{ab, \max \text{power}} = u_{ab, oc}/2$, or from the condition that $R_x = \frac{R_1 R_2}{R_1 + R_2}$, as described above in part 'b').