EI1110 Elkretsanalys, Tentamen TEN1, 2014-10-31 kl 08–13

Hjälpmedel: Ett A4-ark med godtyckligt innehåll (handskriven, datorutskrift, diagram, m.m.).

Tentan har 5 tal i två sektioner: 3 i sektion A (12p), och 2 i sektion B (10p).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för en motstånd, U för en spänningskälla) antas vara kända storheter; och andra markerade storheter (t.ex. strömmen markerad i en motstånd) antas vara okända storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Godkänd tentamen TEN1 kräver:

$$\frac{\max(a, a_{\mathrm{k}})}{A} \geq 40\% \qquad \& \qquad \frac{b}{B} \geq 40\% \qquad \& \qquad \frac{\max(a, a_{\mathrm{k}}) + b}{A+B} \geq 50\%$$

där A=12 och B=10 är de maximala möjliga poängen från sektionerna A och B, a och b är poängen man fick i dessa respektive sektioner i tentan, och a_{k} är poängen man fick från kontrollskrivning KS1 vilken motsvarar tentans sektion A.

Betyget räknas från summan över båda sektioner, igen med bästa av sektion A och KS1, $\frac{\max(a,a_{\mathrm{k}})+b}{A+B}$.
Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).
I vissa gränsfall där betyget är lite under 50%, eller bara en av sektionerna är underkänd trots 50% eller bättre överallt, kommer betyget 'Fx' registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Använd kvarstående tid för att kolla på svaren. Lycka till!

Nathaniel Taylor

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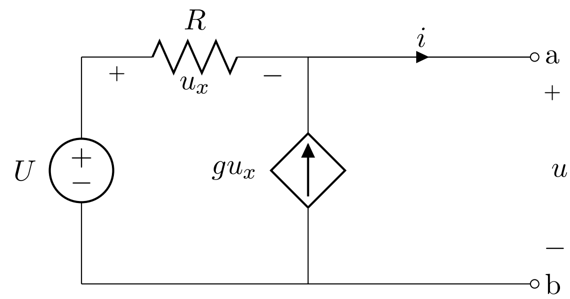
1) [4p]

a) [3p] Bestäm Theveninekvivalenten av kretsen här, med avseende till polerna 'a' och 'b'.

b) [1p] Vilken ström i resulterar i den största effekten *från* kretsen här, ut till en krets eller komponent som kopplas mellan polerna 'a' och 'b'?



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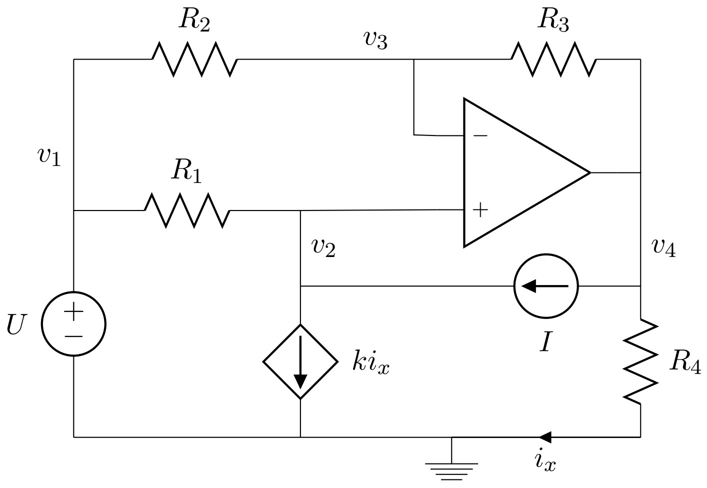
2) [4p]

Skriv ekvationer som skulle kunna lösas för att finna de markerade potentialerna v_1, v_2, v_3, v_4 som funktioner av de givna komponentvärderna.

Du **måste inte** lösa eller förenkla dina ekvationer.

Använd helst en systematisk metod, för att försäkra tillräckliga ekvationer utan onödigt arbete.



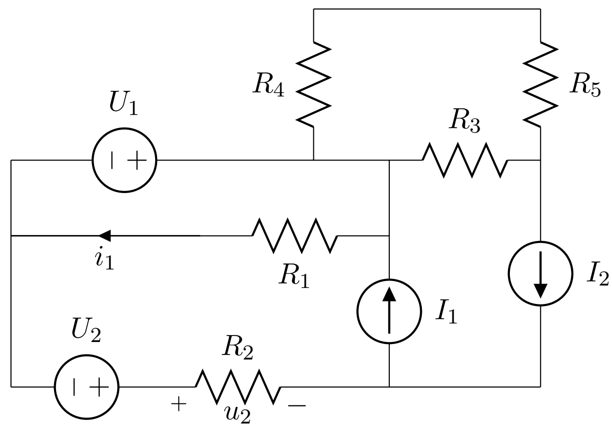
3) [4p]
Bestäm de följande:

a) [1p] Strömmen i_1 (genom R_1).

b) [1p] Spänningen u_2 (över R_2).

c) [1p] Effekten leverad till R_3.

d) [1p] Effekten leverad från källan I_2.



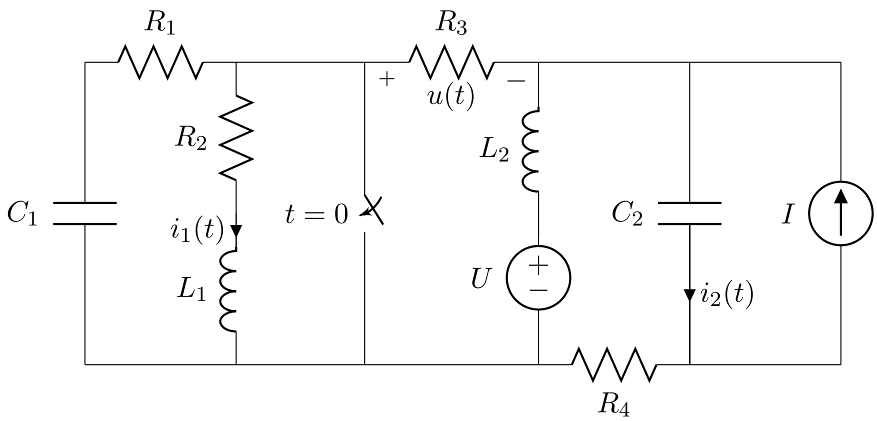
4) [5p]
Bestäm de följande:

a) [1p] $i_2(0^-)$

b) [1p] $i_1(0^+)$

c) [1p] $i_1(\infty)$

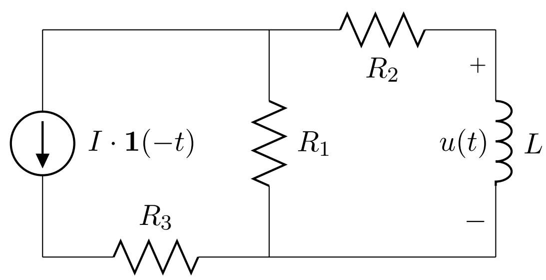
d) [2p] $u(0^+)$



5) [5p]

Bestäm $u(t)$ för tider $t > 0$.

Enhetsstegfunktionen är $\mathbf{1}()$.
Observera minustecken i $\mathbf{1}(-t)$.
(Antag jämvikt innan steget, d.v.s. vid $t = 0^-$.)



Solutions, EI1110 TEN1 2014-10-31

1)

a) The marked voltage across resistor R is $u_x = U - u$.
This determines by Ohm's law the current through the resistor.
It also determines the current in the controlled source, as u_x is the controlling variable.
By KCL on these two currents, the output current i is seen to be

$$i = (U - u)\left(\frac{1}{R} + g\right).$$

The Thevenin equivalent can be directly found from this equation, by rearranging it and comparing terms with the u-i equation of Thevenin source,

$$u \quad = \quad U - \frac{R}{1+gR} i \quad = \quad U_\mathrm{T} + iR_\mathrm{T}.$$

This shows $U_\mathrm{T} = U$ and $R_\mathrm{T} = \frac{R}{1+gR}$.

An alternative method that would also work well is to find the open-circuit voltage and short-circuit current. In short-circuit, $u = 0$ and therefore $u_x = U$ so $i_\mathrm{sc} = U\left(\frac{1}{R} + g\right)$. In open-circuit, KCL with $i = 0$ demands that $u_x\left(\frac{1}{R} + g\right) = 0$ which implies $u_x = 0$, so $u_\mathrm{oc} = U$. The Thevenin resistance is then $u_\mathrm{oc}/i_\mathrm{sc}$. Finding the resistance directly, by 'setting independent sources to zero' is in this case (with a dependent source) *not* an easier alternative to calculating i_sc: proper inclusion of the dependent source requires as much work as just doing the full short/open-circuit method.

b) The maximum possible power is delivered from the two-terminal circuit when it supplies half of its short-circuit current, or (equivalently) when the current is enough to reduce the circuit's terminal voltage to half of its open-circuit value.

The short-circuit current was already calculated in part 'a)'.
For maximum power, we need a current $i_\mathrm{maxP} = \frac{U}{2}\left(\frac{1}{R} + g\right)$.

2) Two solution methods are shown, and a numerical check is made.

Extended nodal analysis ("the simple way")

Let's define the unknown current in voltage source U as i_α, going into the source's + terminal.

KCL (outgoing currents) at all nodes except ground:

$$\mathrm{KCL}(1): \quad 0 \quad = \quad i_\alpha + \frac{v_1 - v_2}{R_1} + \frac{v_1 - v_3}{R_2} \tag{1}$$

$$\mathrm{KCL}(2): \quad 0 \quad = \quad ki_x - I + \frac{v_2 - v_1}{R_1} \tag{2}$$

$$\mathrm{KCL}(3): \quad 0 \quad = \quad \frac{v_3 - v_1}{R_2} + \frac{v_3 - v_4}{R_3} \tag{3}$$

$$\mathrm{KCL}(4): \quad 0 \quad = \quad \frac{v_4 - v_3}{R_3} + i_\beta + I + \frac{v_4}{R_4} \tag{4}$$

These are 7 unknowns $(v_1, v_2, v_3, v_4, i_x, i_\alpha, i_\beta)$ in just 4 equations.

Next, include the further information given by the voltage source,

$$v_1 \quad = \quad U, \tag{5}$$

and also define the controlling variable i_x in terms of the existing known or unknown quantities,

$$i_x = \frac{v_4}{R_4}. \tag{6}$$

There is still one equation too few. The opamp has introduced an unknown current i_β in its output, for KCL at node 4. Some further knowledge about the opamp should provide a further equation. We use the assumption of negative feedback and infinite gain, to state that the opamp's two inputs will be forced to have equal potential,

$$v_3 = v_2. \tag{7}$$

(Alternatively, we could approximate this situation by assigning a high but finite gain A to the opamp, then representing it as a voltage-controlled voltage source such that $v_4 = A\,(v_2 - v_3)$; this is what is done to model the opamp with a VCVS in SPICE.)

To solve these equations, we can use a symbolic solver to avoid even the need of rearranging into a matrix form for numeric calculation.

```
% solve the above system of 7 equations in 7 unknowns
s = solve( ...
  '0 = ia + (v1-v2)/R1 + (v1-v3)/R2', ...
  '0 = k*ix - I_ + (v2-v1)/R1', ...
  '0 = (v3-v1)/R2 + (v3-v4)/R3', ...
  '0 = (v4-v3)/R3 + ib + I_ + v4/R4', ...
  'v1 = U', ...
  'ix = v4/R4', ...
  'v3 = v2', ...
  'v1, v2, v3, v4, ix, ia, ib' )
% show the results symbolically
%   for f=fields(s)', f{1}, getfield(s,f{1}), end

% set some numbers for the components, and calculate the result;
% this makes it easy to compare the solution with SPICE
U = 10,  I_ = 0.33,  R1 = 47,  R2 = 68,  R3 = 333,  R4 = 100,  k = 5
%
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))) ); end
%
 ia: -0.019350
 ib: -0.390380
 ix:  0.068288
 v1: 10.000000
 v2:  9.462246
 v3:  9.462246
 v4:  6.828832
```

Alternative: Supernode method

KCL is done at each node (or supernode group) apart from the ground node. Notice, in this circuit, that the ground node is part of a supernode: this ground supernode includes node 1 and node 4 (because of the opamp output). So just nodes 2 and 3 need KCL equations.

In order to get a small but solvable set of equations, we can express the marked (but unknown) i_x in terms of other quantities, $i_x = \frac{v_4}{R_4}$. The potential v_1 can be directly substituted as U. The output v_4 of the ideal opamp cannot be directly expressed in terms of the VCVS in the opamp model: instead, we have to take advantage of the principle that $v_+ - v_-$, which allows us to substitute v_3 for v_2 or vice versa. With these substitutions, the KCL equations at the non-ground nodes are

$$\begin{aligned} \text{KCL}(2): \quad 0 &= k\frac{v_4}{R_4} - I + \frac{v_3 - U}{R_1} \\ \text{KCL}(3): \quad 0 &= \frac{v_3 - U}{R_2} + \frac{v_3 - v_4}{R_3}. \end{aligned}$$

It is not sufficient to answer with just the above equations, without also saying how to find the remaining two potentials (even if it's obvious),

$$\begin{aligned} \text{known voltage source to known (zero) potential}: \quad v_1 &= U \\ \text{opamp negative feedback assumption}: \quad v_2 &= v_3. \end{aligned}$$

Checking the answers

When we actually care about a result, it is wise to double-check by another route that is as independent as possible. A specific circuit-solver lets us check by a route that is independent of the nodal equations that we wrote.

The following SPICE netlist describes the same circuit.
The component called `E1` models the opamp as a VCVS of 'high' gain, $A = 10^9$.
The component `Vix` is a zeroed voltage source between R_4 and ground, where the circuit's marked current i_x can be measured for use as the controlling variable to CCCS `F1`. This program calculates currents in voltage sources, so it is common to add zeroed voltage sources to measure currents.

```
EI1110_HT14_TEN1Q2
V1    1 0  DC  10.0
I1    4 2  DC   0.33
F1    2 0  Vix  5
E1    4 0  2 3 1e9
R1    1 2       47
R2    3 1       68
R3    4 3      333
R4    4 5      100
Vix   5 0        0
.OP
.PRINT DC V(0) V(1) V(2) V(3) V(4)
.END
```

The venerable program from 1983 (!) solves this tiny, simple, linear dc circuit with the following output.

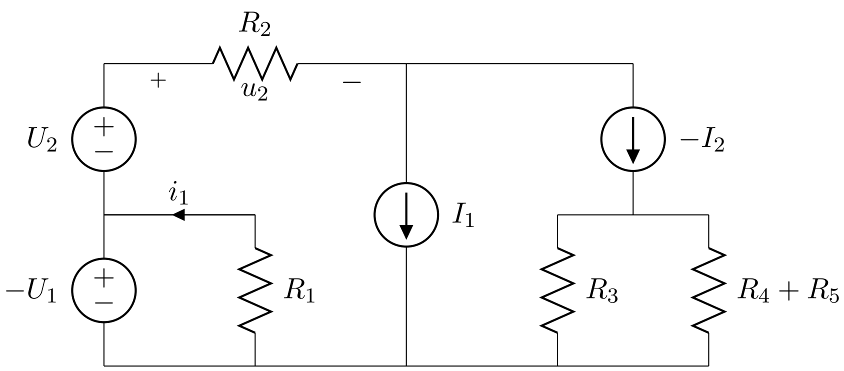
```
 *******10/24/14 ********  spice 2g.6    3/15/83 ********22:48:16*****

 node   voltage     node   voltage     node   voltage     node   voltage
 (  1)   10.0000    (  2)    9.4622    (  3)    9.4622    (  4)    6.8288

 voltage source currents
   name        current
   v1       -1.935E-02
   vix       6.829E-02
```

3)

Re-drawing the circuit may be of help, although it is not obvious that this will always be useful. There is one node, at the bottom of R_4, that clearly has more components connecting to it than any other node: let's put this big node at the bottom of our new diagram.



Then it is important to double-check that all the information from the original diagram matches with our new diagram. The new diagram, above, has combined R_4 and R_5, which means we have lost information about the node between these two ... if we wanted to find the potential at that node, then we would have to look at the original diagram to find whether R_4 should be the upper or lower resistor in our new version. But we don't need to find this potential, so it's ok. The sort of check that one should do thoroughly includes such questions as *does i_1 go towards the node where U_1 and U_2 have their $-$ terminals?* (or $-U_1$ has its $+$ terminal).

a) By noticing that R_1 is parallel with source U_1, we find $i_1 = \frac{U_1}{R_1}$.

b) The current through R_2 is the sum of currents due to the two current sources, which allows its voltage to be calculated: $u_2 = (I_1 - I_2)\, R_2$.

c) The current of I_2 is divided between the $R_4 + R_5$ pair and R_3.
The power dissipation in R_3 is $i_3^2 R_3$ if we define i_3 as the current through this resistor (the direction doesn't matter, as the current is squared to find power — we know a positive resistor's voltage will always be in the direction to cause the current to lose energy).
So, using the current division equation to find i_3, $P_{R_3} = \left(\frac{I_2(R_4+R_5)}{R_3+R_4+R_5}\right)^2 R_3$.

d) This is somewhat long. We have to find the voltage across source I_2, then to multiply this with the source's current to find the delivered power. (It is important to check the definition directions of current and voltage, in order to use the right sign for calculating power.)
The voltage across the source I_2 can be found by KVL around the outside loop of the circuit.
We get $P_{I_2} = I_2\left(U_2 - U_1 + (I_2 - I_1)\, R_2 + I_2 \left(\frac{R_3(R_4+R_5)}{R_3+R_4+R_5}\right)\right)$.

4)

a) $i_2(0^-) = 0$. This is seen from the equilibrium condition for a capacitor, $\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \to i = 0$.

b) $i_1(0^+) = i_1(0^-) = \frac{U}{R_2+R_3}$. This solution uses continuity between $t = 0^-$ and $t = 0^+$ for the current in L_1. In the initial equilibrium at 0^-, resistors R_2 and R_3 are in series and connected directly to voltage source U when we consider that the inductors have zero voltage in equilibrium.

c) $i_1(\infty) = 0$. The closed switch short-circuits the R_2-L_1 branch (you could think of the closed switch as a zeroed voltage source, if that helps!) so the rest of the circuit is irrelevant to i_1. The relevant circuit is just this resistor and inductor connected directly together. The initial current will decay towards zero as time goes on.

d) $u(0^+) = -U\frac{R_3(R_2+R_3+R_4)}{(R_2+R_3)(R_3+R_4)}$. This is definitely the hardest part of this question.
On the left of R_3 is a short-circuit.
On the right is a branch of L_2 and U: before the switch closed, we know by equilibrium and KCL that the current up through this branch was $\frac{U}{R_2+R_3} - I$; the inductor L_2 ensures that this current is continuous, so this whole branch at time $t = 0^+$ looks like a current source of $\frac{U}{R_2+R_3} - I$.
Further to the right are a capacitor and current source in parallel; the capacitor voltage is a continuous variable, so during the short times around switching it behaves like a voltage source, making source I irrelevant. The initial voltage of C_2 is $U + IR_4$ if defined with the reference $+$ upwards. The rightmost three components in the circuit therefore behave as a voltage source $U + IR_4$ and series resistor R_4.

Putting these three simplified branches together, nodal analysis for a potential v in the top node with respect to the bottom is $\frac{v}{R_3} - \frac{U}{R_2+R_3} + I + \frac{v-(U+IR_4)}{R_4} = 0$.

The definition of v that we have chosen in the above is such that $u = -v$, so the solution of v the above nodal equation is the required solution of $u(0^+)$.

5)

The function $\mathbf{1}(-t)$ is 1 for $t \leq 0$ and 0 for $t > 0$. Before this step, the current source therefore has a value I. The equilibrium state of the circuit, in which L can be treated as a short-circuit, is a current division between R_1 and R_2; the resistor R_3 is irrelevant to what happens outside the series branch of I and R_3.

Let's follow the recommendation of dealing with the continuous variable, which is the inductor's current; we can define this as i into the + pole.

The equilibrium current in the inductor before the step is therefore $i(0^-) = -I\frac{R_1}{R_1+R_2}$. Because this is a continuous variable, $i(0^-) = i(0^+)$.

After the step, the current source has zero value, so it is an open circuit. This whole branch can therefore be ignored: the relevant circuit is just the loop of R_1, R_2 and L.

The final equilibrium is $i(\infty) = 0$, as there is no source in the loop. The time-constant is $\tau = \frac{L}{R_1+R_2}$, as the two resistors can be reduced to an equivalent of $R_1 + R_2$. If these two claims feel too much like claims without formal proof, make a Thevenin source of the two resistors, then find the time-constant and equilibrium voltage! Or write and solve the differential equation of the loop.

With the known initial-value, final-value and time-constant, we now find a function based on $\mathrm{e}^{-t/\tau}$ that fits these: $i(t) = -I\frac{R_1}{R_1+R_2}\,\mathrm{e}^{-t/\frac{L}{R_1+R_2}}$, for $t \geq 0$.

As it was in fact the voltage $u(t)$, not the current $i(t)$, that was requested, we must calculate with $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ (after checking that the definition directions of these quantities does not require a negative sign in the equation!).

After simplifying, $u(t) = IR_1\,\mathrm{e}^{-t\Big/\frac{L}{R_1+R_2}} \quad (t > 0)$.
An interesting matter of pedantry is that we should restrict this equation to times *after* zero. Exactly at zero, $u(0) = 0$; but this immediately steps up to $u(0^+) = IR_1$. This situation arises because u is not a continuous variable, and the unit-step function is commonly defined as being equal to 1 at $t = 0$. But this question of step-function definitions is not a practical issue as long as we remember that only the continuous variables can be relied upon to *not* have sudden changes.