

# EI1110 Elkretsanalys, Omtentamen TEN1, 2015-01-09 kl 08–13

**Hjälpmedel:** Ett A4-ark med godtyckligt innehåll (handskriven, datorutskrift, diagram, m.m.).

Tentan har 5 tal i två sektioner: 3 i sektion A (12p), och 2 i sektion B (10p).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex.  $R$  för ett motstånd,  $U$  för en spänningskälla,  $k$  för en beroende källa) antas vara kända storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara okända storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Godkänd tentamen TEN1 kräver:

$$\frac{\max(a, a_k)}{A} \geq 40\% \quad \& \quad \frac{b}{B} \geq 40\% \quad \& \quad \frac{\max(a, a_k) + b}{A + B} \geq 50\%$$

där  $A=12$  och  $B=10$  är de maximala möjliga poängen från sektionerna A och B,  $a$  och  $b$  är poängen man fick i dessa respektive sektioner i tentan, och  $a_k$  är poängen man fick från kontrollskrivning KS1 vilken motsvarar tentans sektion A. Omtentan är ett *alternativ* till ordinarietentan: man kan inte använda t.ex. sektion A från en och sektion B från den andra; i stället räknas den tentamen vilken som helhet ger bästa poäng efter hänsyn till kontrollskrivningen.

Betyget räknas från summan över båda sektioner, igen med bästa av sektion A och KS1,  $\frac{\max(a, a_k) + b}{A + B}$ . Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

I vissa gränfall där betyget är lite under 50%, eller bara en av sektionerna är underkänd trots 50% eller bättre som helhet, kommer betyget 'Fx' registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

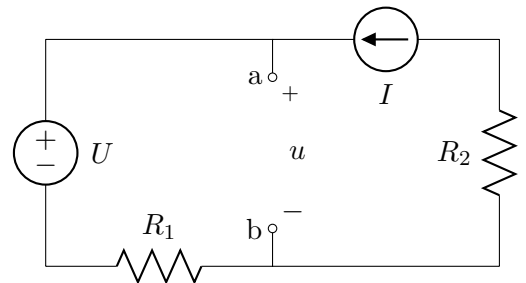
Använd kvarstående tid för att kolla på svaren. Lycka till!

Nathaniel Taylor

1) [4p]

a) [3p] Bestäm Theveninekvivalenten av kretsen här, med avseende på polerna 'a' och 'b'.

b) [1p] Betrakta fallet där polerna 'a' och 'b' kopplas till en extern komponent som är anpassad så att den största möjliga effekten dras från denna krets. Bestäm  $u$  i detta fall.

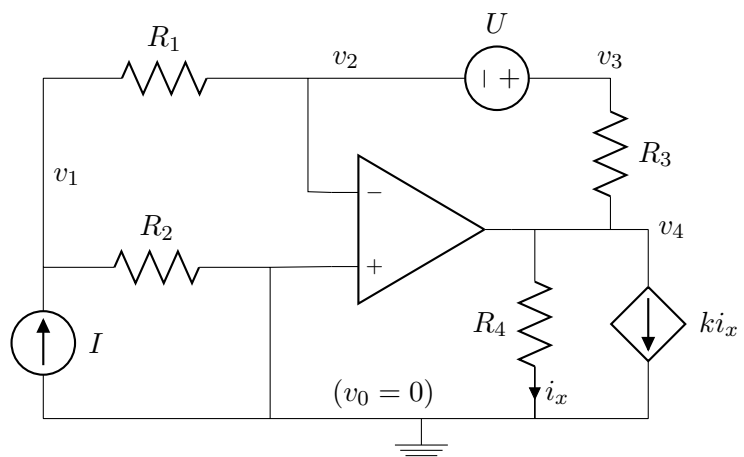


2) [4p]

Skriv ekvationer som skulle kunna lösas för att finna de markerade potentialerna  $v_1, v_2, v_3, v_4$  som funktioner av de givna komponentvärdena.

Du **måste inte** lösa eller förenkla dina ekvationer.

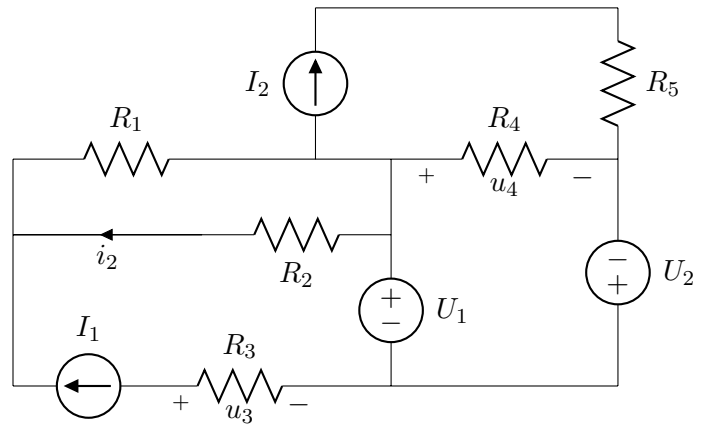
Använd helst en systematisk metod, för att försäkra tillräckliga ekvationer utan onödigt arbete.



3) [4p]

Bestäm de följande:

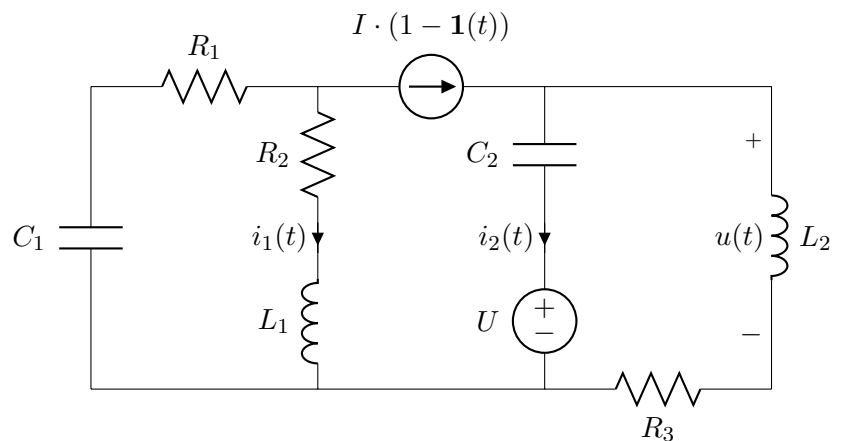
- a) [1p] Spänningen  $u_3$  (över  $R_3$ ).
- b) [1p] Spänningen  $u_4$  (över  $R_4$ ).
- c) [1p] Effekten levererad från källan  $U_2$ .
- d) [1p] Strömmen  $i_2$ .



4) [5p]

Bestäm  $i_1(t)$ ,  $i_2(t)$  och  $u(t)$  vid de följande tiderna (det blir då 9 svar sammanlagd).

- a) [1,5p]  $t = 0^-$
- b) [2p]  $t = 0^+$
- c) [1,5p]  $t \rightarrow \infty$

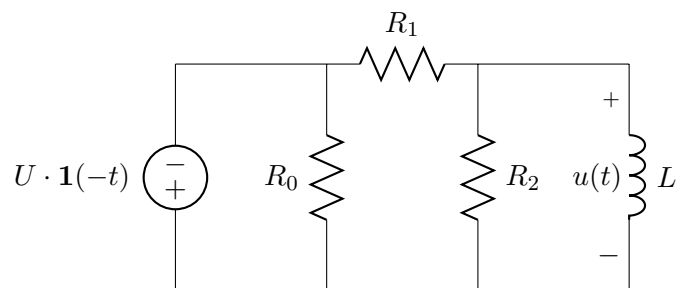


5) [5p]

Bestäm  $u(t)$  för tider  $t > 0$ .

Enhetsstegfunktionen är  $\mathbf{1}()$ .  
Observera minustecken i  $\mathbf{1}(-t)$ .

(Antag jämvikt innan steget, d.v.s. vid  $t = 0^-$ .)



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## Solutions, EI1110 TEN1 re-exam, 2015-01-09

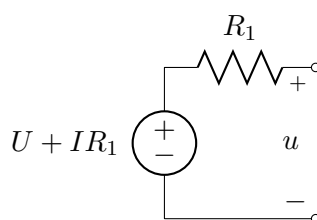
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1)

a) To find the Thevenin equivalent we need to find the open-circuit voltage and the source resistance. There are several ways to do this.

This circuit has no dependent source, so it is probably easiest to find the source resistance directly, by setting all the independent sources to zero: the branch at the right can then be ignored (current-source is open-circuit), and the branch on the left is just  $R_1$  (voltage-source is short-circuit). From this, the resistance seen between terminals ‘a’ and ‘b’ is simply  $R_T = R_1$ .

When the terminals ‘a’ and ‘b’ are open-circuit, all the current from the source  $I$  must travel round the whole loop. The voltage across  $R_1$  is therefore known as  $IR_1$ , and so – taking careful note of the direction of this voltage – we see that  $u_{oc} = U_T = U + IR_1$ .



b) Maximum power transfer is when the output voltage is half of the open-circuit voltage; hence,  $u_{\max\text{pow}} = \frac{U + IR_1}{2}$ .

2) Two solution methods are shown, and a numerical check is made.

### Extended nodal analysis (“the simple way”)

The current in voltage source  $U$  is unknown: let’s define it as  $i_\alpha$ , going into the source’s + terminal. The current at the opamp output is also unknown (the opamp’s output can be regarded as a voltage source whose other side is connected to the ground node); let’s define this current as  $i_\beta$  coming out from the opamp into node 4.

Now write KCL (outgoing currents) at all nodes except ground:

$$\text{KCL(1): } 0 = -I + \frac{v_1 - v_2}{R_1} + \frac{v_1}{R_2} \quad (1)$$

$$\text{KCL(2): } 0 = \frac{v_2 - v_1}{R_1} - i_\alpha + 0 \quad (2)$$

$$\text{KCL(3): } 0 = i_\alpha + \frac{v_3 - v_4}{R_3} \quad (3)$$

$$\text{KCL(4): } 0 = -i_\beta + \frac{v_4 - v_3}{R_3} + \frac{v_4}{R_4} + ki_x \quad (4)$$

These are 7 unknowns ( $v_1, v_2, v_3, v_4, i_x, i_\alpha, i_\beta$ ) in just 4 equations.

Next, include the further information given by the voltage source,

$$v_3 = v_2 + U, \quad (5)$$

and also define the controlling variable  $i_x$  in terms of the existing known or unknown quantities,

$$i_x = \frac{v_4}{R_4}. \quad (6)$$

There is still one equation too few. It's the opamp's fault that there's an unknown current  $i_\beta$  in its output. Some further knowledge about the opamp should provide the further equation that we need. We use the assumption of negative feedback and infinite gain, to state that the opamp's two inputs will be forced to have equal potential,

$$v_+ = v_-, \quad v_2 = 0 \quad (7)$$

(Alternatively, we could approximate this situation by assigning a high but finite gain  $A$  to the opamp, then representing it as a voltage-controlled voltage source such that  $v_4 = A(0 - v_2)$ ; this is what is done if modelling an opamp with a VCVS in SPICE.)

To solve these equations, we can use a symbolic solver to avoid even the need of rearranging into a matrix form for numeric calculation.

```
% solve the above system of 7 equations in 7 unknowns
s = solve( ...
    '0 = -I_ + (v1-v2)/R1 + v1/R2', ...
    '0 = (v2-v1)/R1 - ia', ...
    '0 = ia + (v3-v4)/R3', ...
    '0 = -ib + (v4-v3)/R3 + v4/R4 + k*ix', ...
    'v3 = v2 + U', ...
    'ix = v4/R4', ...
    'v2 = 0', ...
    'v1, v2, v3, v4, ix, ia, ib' )
% show the results symbolically
% for f=fields(s)', f{1}, getfield(s,f{1}), end
v1 = (I_*R1*R2)/(R1 + R2)
v2 = 0
v3 = U
v4 = (R1*U + R2*U - I_*R2*R3)/(R1 + R2)
ia = -(I_*R2)/(R1 + R2)
ib = (R1*U + R2*U - I_*R2*R3 - I_*R2*R4 + R1*U*k + R2*U*k - I_*R2*R3*k)/(R4*(R1 + R2))
ix = (R1*U + R2*U - I_*R2*R3)/(R4*(R1 + R2))

% set some numbers for the components, and calculate the result;
% this makes it easy to compare the solution with SPICE
U = 10, I_ = 0.33, R1 = 47, R2 = 68, R3 = 333, R4 = 100, k = 5
%
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))) ); end
%
v1: 9.171130
v2: 0.000000
v3: 10.000000
v4: -54.978435
ia: -0.195130
ib: -3.493837
ix: -0.549784
```

### Alternative: Supernode method

KCL is done at each node (or supernode group) apart from the ground node.

In this circuit, the ground node and node 4 form a supernode together because of being connected through the opamp. This means that no KCL is needed on either of these nodes. The *unknown* voltage of the opamp means that we cannot just represent  $v_4$  directly (in the way we would if node 4 were connected to ground by an independent voltage source like ' $U_1$ ' or by a dependent voltage source like  $hi_y$ ). Instead, we can keep the unknown  $v_4$  when we need this potential for writing the KCL equation for node 3, and we can then use the  $v_+ = v_-$  assumption to provide another equation to make the system solvable.

Nodes 2 and 3 also are a supernode, as they are connected through source  $U$ .

Components  $R_4$  and  $ki_x$  are connected between parts of the same supernode, so they are not relevant to any KCL that is needed in the supernode method.

Only two KCL equations are therefore needed: one for node 1, and the other for the supernode (2+3).

$$\begin{aligned} \text{KCL}(1) : \quad 0 &= -I + \frac{v_1 - v_2}{R_1} + \frac{v_1}{R_2} \\ \text{KCL}(2+3) : \quad 0 &= \frac{v_2 - v_1}{R_1} + \frac{v_2 + U - v_4}{R_3}. \end{aligned}$$

It is not sufficient to answer with just the above equations, without also saying how to find the remaining two potentials (even if it's obvious),

$$\begin{aligned} \text{known voltage source to known (zero) potential} : \quad v_3 &= v_2 + U \\ \text{opamp negative feedback assumption} : \quad v_2 &= 0. \end{aligned}$$

### Checking the answers

When we really care about a solution, perhaps because we're going to base some later work on it, then it's wise to double-check by another route that is as independent as possible. A specific circuit-solver lets us check by a route that is independent of the nodal equations that we wrote.

The following SPICE netlist describes the same circuit.

The component called E1 models the opamp as a VCVS of 'high' gain,  $A = 10^9$ .

The component Vix is a zeroed voltage source between an extra node (5) introduced at the bottom of  $R_4$ , and ground; this is the way in which current  $i_x$  can be measured in order to use it as the controlling variable to the VCCS F1. SPICE calculates currents in voltage sources, so it is common to add zeroed voltage sources to measure currents.

```

EI1110_HT14_OMTEN1Q2
V1 3 2 DC 10.0
I1 0 1 DC 0.33
F1 4 0 Vix 5
E1 4 0 0 2 1e9
R1 1 2 47
R2 1 0 68
R3 4 3 333
R4 4 5 100
Vix 5 0 0
.OP
.PRINT DC V(0) V(1) V(2) V(3) V(4)
.END

```

The venerable program from 1983 (!) solves this tiny, simple, linear dc circuit with the following output.

```

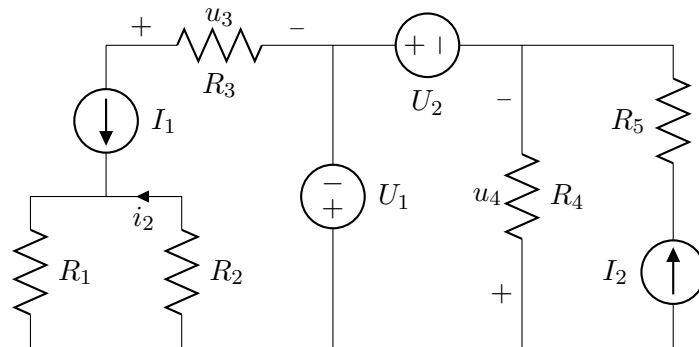
v1 = 9.1711
v2 = 0.0000
v3 = 10.0000
v4 = -54.9784

(voltage-source currents:
v1: -1.951E-01
vix: -5.498E-01
controlled sources:
E1: v-source -54.978
E1: i-source 3.49E+00
F1: i-source -2.75E+00
)

```

3)

Re-drawing the circuit may be of help, although it is not obvious that this will always be useful. There is one node, at the left of  $R_4$ , that clearly has more components connecting to it than any other node: let's put this big node at the bottom of our new diagram.



Then it is important to double-check that all the information from the original diagram matches with our new diagram. Check things like the number of each type of component, the nodes, which components connect between the nodes, and the directions of the components and any marked quantities such as  $i_2$ . Now, based this or another re-drawn diagram, we find the requested quantities.

a)  $R_3$  is series with source  $I_1$ . By KCL, the current in  $R_3$  must be  $I_1$ .  
By Ohm's law, taking into account the directions,  $u_3 = -I_1 R_3$ .

b) By KVL around the loop of  $U_1$ ,  $u_4$ ,  $U_2$ , we have  $U_1 - u_4 + U_2 = 0$ , hence  $u_4 = U_1 + U_2$ .

c) The power *from* source  $U_2$  is the current out of its '+' terminal, multiplied by the source's value  $U_2$ . From sub-question 'b', we can use Ohm's law to find that the current flowing in resistor  $R_4$  towards source  $U_2$  is  $\frac{U_1+U_2}{R_4}$ . The current in  $R_5$  is determined by the source  $I_2$ , in series with it.

By KCL in the node where  $R_4$ ,  $R_5$  and  $U_2$  connect, the current coming into the '-' terminal of source  $U_2$  is therefore  $I_2 + \frac{U_1+U_2}{R_4}$ .

This must be the same as the current coming out of the '+' terminal of source  $U_2$ .

Therefore,  $P_{U_2, \text{out}} = U_2 I_2 + U_2 \frac{U_1+U_2}{R_4}$ .

d) The resistors  $R_1$  and  $R_2$  are in parallel (check!).

The current through the pair is determined by source  $I_1$ .

Current division is therefore useful here, being careful about the relative directions,  $i_2 = -I_1 \frac{R_1}{R_1+R_2}$ .

4)

a) Current source gives  $I$ . Circuit is in equilibrium.

$i_1(0^-) = -I$   $C_1$  behaves as an open-circuit, when in equilibrium; then use KCL above  $R_2$

$i_2(0^-) = 0$  similar principle to the above:  $C_2$  behaves as an open-circuit

$u(0^-) = 0$   $L_2$  behaves as a short-circuit, in equilibrium

b) Current source has just now changed to 0. Circuit is not in equilibrium.

$i_1(0^+) = -I$  continuity of current in  $L_1$

$i_2(0^+) = -I$  continuity of current in  $L_2$ , and KCL above  $C_2$

$u(0^+) = 0$  continuity of voltage on  $C_2$ , continuity of current in  $L_2$ , then KVL around their loop

c) Current source is still 0. Circuit is in equilibrium.

$i_1(\infty) = 0$  the same reasoning as in part 'a)', but the current-source is now zero

$i_2(\infty) = 0$  same as part 'a)': equilibrium in  $C_2$  means no current

$u(\infty) = 0$  see part 'a)': equilibrium in  $L_2$  means no voltage

5)

The function  $\mathbf{1}(-t)$  is almost the same as the  $(1 - \mathbf{1}(t))$  from Question 4 (the difference is only in what happens at exactly  $t = 0$ ). So the voltage source has a value  $U$  before time  $t = 0$ , and is zero after this. It is therefore like a short circuit during the period  $t > 0$  for which we need to find  $u(t)$ ; the only thing driving the circuit in that period is the energy that was stored in the inductor from before the step. This energy corresponds to a current: it will die away, as the current has to pass through resistors, which will consume energy.

The initial condition of the inductor can be found from the assumption of equilibrium at  $t = 0^-$ . To find this state, the inductor can be represented as a short-circuit, and the voltage source as  $U$ . Let's define the inductor's current as  $i(t)$ , going downwards so that  $u(t) = L \frac{di(t)}{dt}$ . Resistors  $R_0$  and  $R_2$  are irrelevant, as they are in parallel with a voltage source or a short circuit respectively.

Thus,  $i(0^-) = -U/R_1$ ; the negation is because of the direction of the source.

After  $t = 0$ , when the source has become zero, the  $R_0$  is parallel with a short-circuit so it can still be ignored. (Be careful:  $R_2$  cannot be assumed to be parallel with a short-circuit, since the inductor is only like a short-circuit if we can assume it to be in an equilibrium state.) In this period  $t > 0$ , the inductor is therefore connected to the parallel combination of  $R_1$  and  $R_2$ , i.e.  $\frac{R_1 R_2}{R_1 + R_2}$ .

This is then a simple question of an inductor and resistor connected together in a loop, with a known initial condition in the inductor.

The equilibrium is  $i(\infty) = 0$ .

The initial condition is  $i(0^+) = i(0^-) = -U/R_1$ .

The time-constant is  $\tau = L \frac{R_1 + R_2}{R_1 R_2}$ .

Putting these together into the known 'shape' of a first-order ODE solution, we get

$$i(t) = -\frac{U}{R_1} e^{-t \frac{R_1 R_2}{L(R_1 + R_2)}},$$

which can be shown from the principle of  $y(t) = y_\infty - (y_0 - y_\infty) e^{-t/\tau}$ .

It was in fact the *voltage* across the inductor that was to be found:  $u(t) = L \frac{di(t)}{dt}$ . The solution for current must therefore be differentiated and multiplied by  $L$ ,

$$u(t) = -\frac{U}{R_1} L \frac{-R_1 R_2}{L(R_1 + R_2)} e^{-t \frac{R_1 R_2}{L(R_1 + R_2)}},$$

so  $u(t) = \frac{U R_2}{R_1 + R_2} e^{-t \frac{R_1 R_2}{L(R_1 + R_2)}}$ .

Showing this more formally by ODE solution is left 'as an exercise for the reader'. The only requirement for the exam is that you get the right answer, with some explanation that indicates a reasonable method.

[A strange substitution of  $I$  instead of  $U/R_1$  was pointed out in 2016-10 (thanks!). This is now corrected.]