KTH EI1120 Elkretsanalys (CENMI), Tentamen 2015-03-17 kl 08–13

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, . . .).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_{k} vara poängen man fick från kontrollskrivning KS1, och b_{k} poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a,a_{\mathrm{k}})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b,b_{\mathrm{k}})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a,a_{\mathrm{k}})+\max(b,b_{\mathrm{k}})+c+h}{A+B+C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Examinator: Nathaniel Taylor

Del A. Likström

1) [4p]

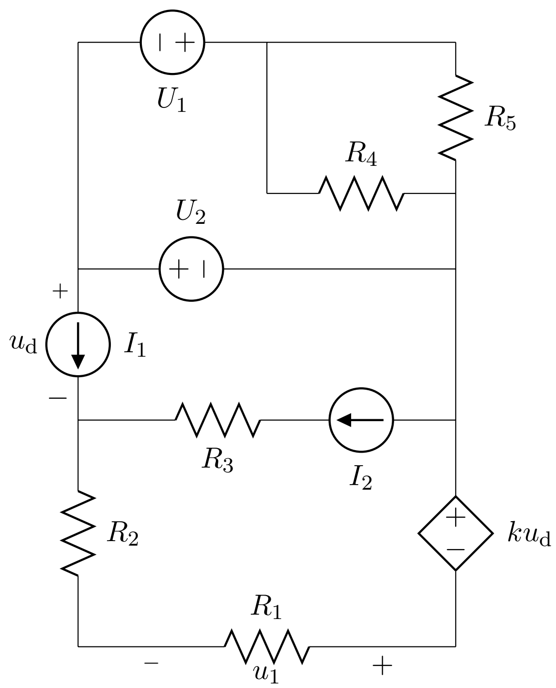
Bestäm de följande:

a) [1p] Effekten levererat till R_3.

b) [1p] Effekten levererat till R_4.

c) [1p] Spänningen u_1 över R_1.

d) [1p] Effekten levererat från källan I_1.

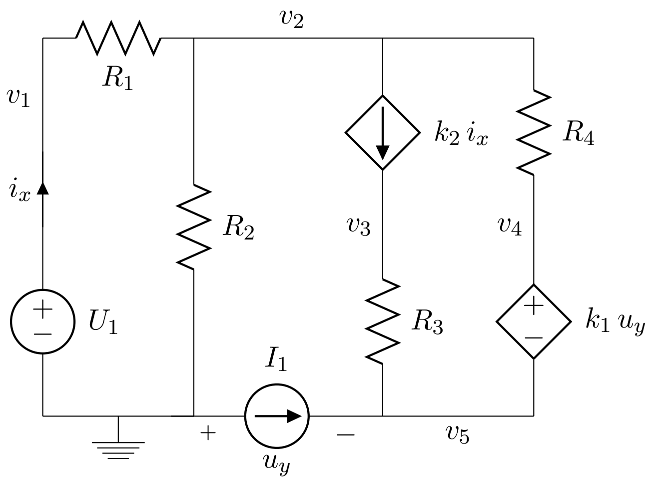


2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1, v_2, v_3, v_4, v_5.

Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.

Som vanligt är det komponentvärdena R_1, I_1, k_1 o.s.v. som är kända, medan de markerade storheterna v_1, i_x, o.s.v. är okända.



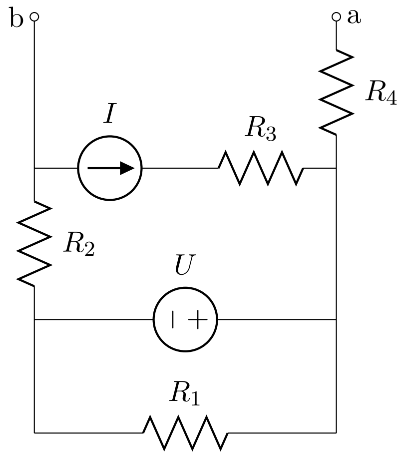
3) [4p]

Låt $R_1 = R_2 = R_3 = R_4 = R$, och $I = U/R$.
Bara U och R är kända storheter.
Svaren borde därför *inte* uttryckas i R_1, I, o.s.v.

a) [2p] Bestäm Theveninekvivalenten av kretsen, med avseende på polerna 'a' och 'b'. Rita upp ekvivalenten inklusive polerna.

b) [1p] Hur mycket är maximaleffekten som kan levereras från kretsen mellan polerna a-b?

c) [1p] En spänningskälla U_x är nu ansluten till kretsen, med sin +-pol till kretsens pol a, och − till pol b. Vilket värde måste spänningen U_x ha för att den maximala möjliga effekten dras från kretsen till källan?

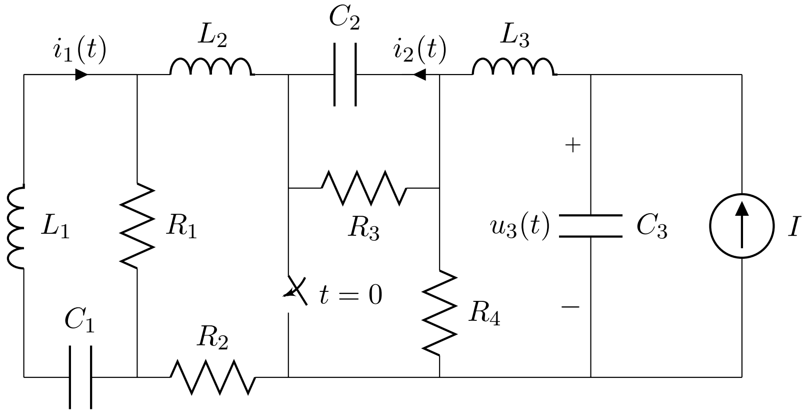


Del B. Transient

4) [5p]

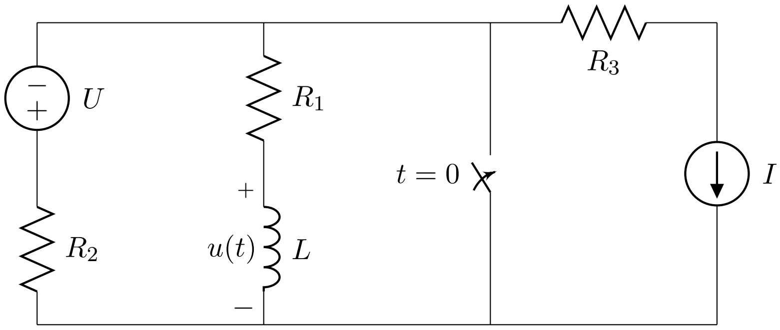
a) [2p] Betrakta jämvikten när $t \to \infty$. Bestäm $i_1(\infty)$ och $u_3(\infty)$.

b) [3p] Betrakta tiden $t = 0^+$, d.v.s. direkt efter brytaren slås på.
Bestäm $i_1(0^+)$, $u_3(0^+)$ och $i_2(0^+)$.



5) [5p]

Bestäm $u(t)$, för $t > 0$.



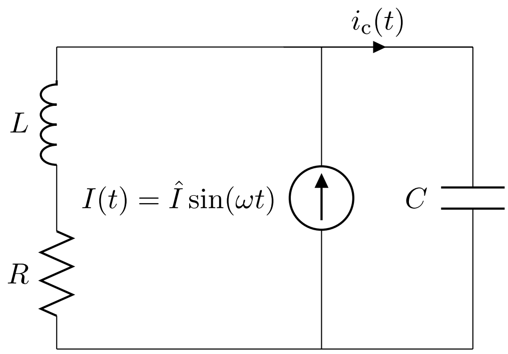
Del C. Växelström

6) [4p]

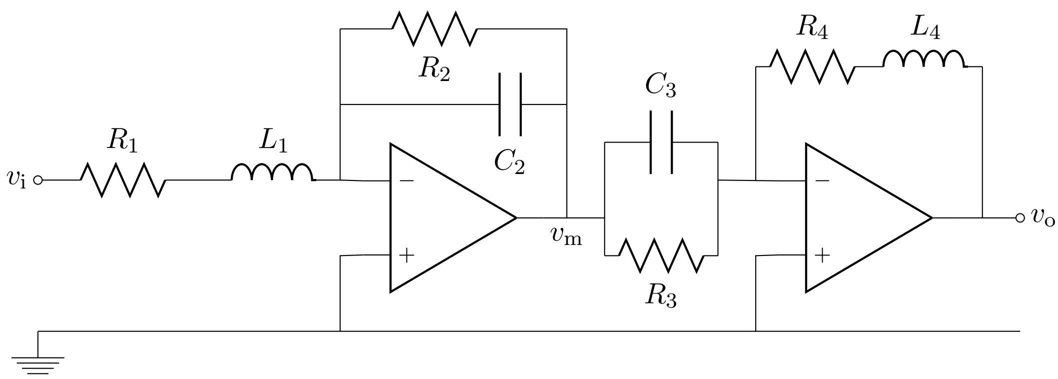
a) [2p] Bestäm $i_c(t)$.

b) [1p] Bestäm den genomsnittliga effekten ('aktiveffekt' i växelströmsterminologi) förbrukat av motståndet R.

c) [1p] Vilken genomsnittlig effekt levereras av strömkällan $I(t)$ om en spänningskälla $\hat{U}\cos(3\omega t + \pi/4)$ nu är seriekopplad mellan R och L med pluspolen uppåt.



7) [5p]



a) [2p] Bestäm nätverksfunktionen $H(\omega) = \frac{v_o(\omega)}{v_i(\omega)}$ av kretsen ovan.
Det kan hjälpa att räkna i två steg, med hjälp av den (okända) potentialen v_m i mitten av kretsen.

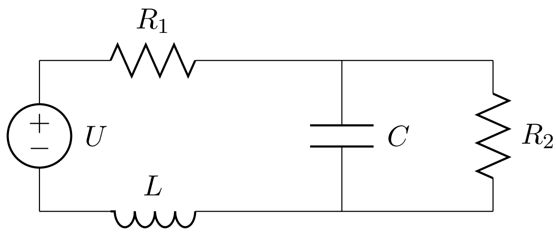
b) [1p] Visa att svaret till deltal 'a' kan skrivas i den följande formen,

$$H(\omega) = k \cdot \frac{(1 + j\omega/\omega_\beta)(1 + j\omega/\omega_\gamma)}{(1 + j\omega/\omega_\alpha)(1 + j\omega/\omega_\delta)},$$

där k och $\omega_{\alpha,\beta,\gamma,\delta}$ är positiva reella tal (på antagandet att komponentvärdena också är det).
Det räcker att uttrycka de fem konstanterna i de kända komponentvärdena.

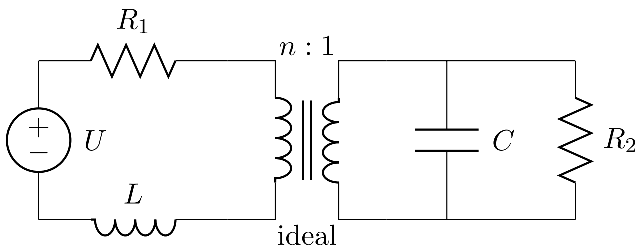
c) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'.
Anta att $100\,\omega_\alpha = \omega_\beta = \omega_\gamma = \frac{1}{100}\,\omega_\delta$, och att $k = 10$.
Markera viktiga punkter och lutningar.

8) [5p]



Deltal ’a’ och ’b’ betraktar bara den övre kretsen.

Källan ger en växelspänning med effektivvärdet U och vinkelfrekvens ω. Komponentvärdena R_2 och C är okända, och får bestämmas av oss. De andra komponenterna har fasta, kända värden.



a) [3p] Välj R_2 och C för att maximera effekten som levereras till motståndet R_2.

b) [1p] Vad är den maximala effekten levererad till R_2 (i fallet av deltal ’a’)? Uttryck den som en funktion av de kända variablerna.

c) [1p] Betrakta nu den nedre kretsen. Skillnaden är bara att en transformator används i mitten av kretsen, i stället för direktkoppling. Lös deltal ’a’ igen, men för den nedre kretsen. Betrakta n som känd.

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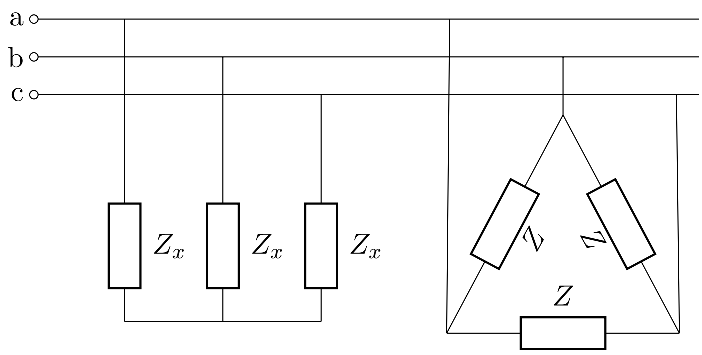
9) [4p]



Polerna a,b,c visar en anslutning till en balanserad trefas spänningskälla, av vinkelfrekvens ω.
Källans huvudspänning är U, d.v.s. $|u_{\mathrm{ab}}| = |u_{\mathrm{bc}}| = |u_{\mathrm{ca}}| = U$. Som vanligt när det gäller elkraft, är det ett effektivvärde.

Varje impedans Z representerar ett motstånd R och en spole L, *parallell*kopplade. Värdet av Z_x ska bestämmas. De kända storheterna är U, ω, R och L.

a) [2p] Vilken aktiv effekt och reaktiv effekt förbrukas av delta-lasten (de tre impedanserna Z)?

b) [1p] Bestäm en komponent (spole, kondensator, eller motstånd) för Z_x, och dess värde, för att effektfaktorn (PF) av alla sex impedanser, sett från källan vid polerna a,b,c, blir 1.

c) [1p] Låt impedanserna Z modellera en maskin som säljs till en kund som kräver att effektfaktorn sett på maskinens poler a,b,c ska vara lägst $1/\sqrt{2}$ induktiv (cirka 0,71). Tyvärr har vi gjort maskinen sådan att $\omega L = R/\sqrt{3}$. Vilken effektfaktor har Δ-lasten i så fall?
Man skulle kunna välja en komponent till Z_x enligt lösningen till deltal ’b’, då $\text{PF} = 1$ fyller kravet. Men man kan i stället välja samma slags komponent med ett mindre värde som ger bara den nödvändiga effektfaktorkompenseringen för att uppfylla kravet: vilket värde behövs i så fall?
Alternativt, skulle man kunna uppfylla kravet genom att välja en annan slags komponent, av de tre möjligheterna listade i deltal ’b’. Ett sådant val är bara ’juridiskt skoj’ med en otillräckligt begränsad specifikation av kravet ... men i alla fall: vilken komponent kan den vara, och vilket värde behövs för att uppfylla kravet om PF, med minimal skenbar effekt i Z_x?

Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1120, VT15, 2015-03-17)

Q1

a) Power into R_3 is $I_2^2 R_3$ (it's in series with the current source).

b) Power into R_4 is $\dfrac{(U_1+U_2)^2}{R_4}$ (it's directly connected to the pair of voltage sources).

c) Voltage across R_1 is $u_1 = -R_1(I_1+I_2)$ (by KCL with the two current-sources).

d) This is definitely the hardest question!
The power out from source I_1 is $-I_1 u_\text{d}$, but u_d is not known.
It can only be solved by considering the (relevant parts of the) rest of the circuit. Consider KVL around the smallest loop that includes source I_1: then $u_\text{d} = I_2 R_3 + ? + U_2$, where '?' is the voltage across the *other* current-source! We could do another KVL in the lower loop, to get another expression including the same '?', then substitute it into the above. But it's easier just to take KVL around a bigger loop that doesn't include the other current source at all. Hence,

$$u_\text{d} = U_2 + k u_\text{d} - (R_1+R_2)(I_1+I_2),$$

whence

$$u_\text{d} = \frac{U_2 - (I_1+I_2)(R_1+R_2)}{1-k},$$

so the power is

$$P_{I_1(\text{out})} = -I_1 u_\text{d} = \frac{I_1(I_1+I_2)(R_1+R_2) - I_1 U_2}{1-k}.$$

Q2 As usual, we show several alternative solutions here: many more are possible.

Extended nodal analysis ("the simple way")

Let's define the unknown currents in the voltage sources, with the positive direction going into the source's + terminal: i_α in source U_1, and i_β in source $k_1 u_y$.

KCL (outgoing currents) at all nodes except ground:

$$\begin{aligned}
\text{KCL}(1): \quad 0 &= i_\alpha + \frac{v_1 - v_2}{R_1} && (1)\\
\text{KCL}(2): \quad 0 &= \frac{v_2 - v_1}{R_1} + \frac{v_2}{R_2} + k_2 i_x + \frac{v_2 - v_4}{R_4} && (2)\\
\text{KCL}(3): \quad 0 &= \frac{v_3 - v_5}{R_3} - k_2 i_x && (3)\\
\text{KCL}(4): \quad 0 &= \frac{v_4 - v_2}{R_4} + i_\beta && (4)\\
\text{KCL}(5): \quad 0 &= -I_1 + \frac{v_5 - v_3}{R_3} - i_\beta && (5)
\end{aligned}$$

Now there are 9 unknowns $(v_1, v_2, v_3, v_4, v_5, i_\alpha, i_\beta, i_x, u_y)$, but only 5 equations. So we add the further information that the voltage sources provide,

$$\begin{aligned}
v_1 - 0 &= U_1 && (6)\\
v_4 - v_5 &= k_1 u_y && (7)
\end{aligned}$$

and also define the controlling variables i_x and u_y in terms of the existing known or unknown quantities,

$$\begin{aligned}
i_x &= -i_\alpha && (8)\\
u_y &= 0 - v_5. && (9)
\end{aligned}$$

There are now 9 equations in 9 unknowns. The systematic way in which this was done is important! We can be confident that the equations are all providing useful information, i.e. that they are not linearly dependent.

There are various possibilities for writing the above. For example, i_x is marked in the diagram between source U_1 and resistor R_1, in the 'trivial node' where they join. It could therefore equally well have been written as $i_x = \frac{v_1 - v_2}{R_1}$. The choice makes no difference to the result: the algebra already 'knows' from equation (1) these two statements are equivalent. Or, we could have noticed from the start that the current in U_1 is already defined as i_x, out of the + pole, then we would have had just 8 variables and could have avoided the equation (8).

For comparison between the symbolic solution from our nodal equations, and the numeric SPICE circuit solution, we can assign some numeric values to the component values:
`U1=15, I1=0.5, R1=12, R2=5, R3=8, R4=3, k1=2, k2=1.5`

The circuit can then be described by the following "netlist" for solving in SPICE:

```
EI1120_VT15_TEN
V1    1 0  DC  15.0
I1    0 5  DC   0.5
R1    1 2      12.0
R2    2 0       5.0
F1    2 3  V1  -1.5
E1    4 5  0 5  2.0
R3    3 5       8.0
R4    2 4       3.0
.OP
.PRINT DC V(0) V(1) V(2) V(3) V(4) V(5)
.END
```

which results in the following output (reformated):

```
POTENTIALS
   v1:   15.0000
   v2:    6.1765
   v3:   -2.1618
   v4:   10.9853
   v5:  -10.9853
VSRC
    i:    -7.353E-01  (ia)
VCVS
    v:    21.971
    i:    -1.60E+00
CCCS
    i:     1.10E+00
```

This can be conveniently compared to the earlier set of symbolic equations, by solving those equations in (e.g.) Matlab symbolic toolbox and then substituting the same numeric values as were used in the SPICE solution.

```
s = solve( ...
  '0 =  ia + (v1-v2)/R1', ...
  '0 =  (v2-v1)/R1 + v2/R2  + k2*ix + (v2-v4)/R4', ...
  '0 =  (v3-v5)/R3 - k2*ix', ...
  '0 =  (v4-v2)/R4 + ib', ...
  '0 =  -I1 + (v5-v3)/R3 - ib', ...
  'v1 - 0 = U1', ...
  'v4 - v5 = k1*uy', ...
  'ix = -ia', ...
  'uy = 0 - v5', ...
  'v1,v2,v3,v4,v5,ia,ib,ix,uy' )

%% set numeric values for comparison with the SPICE solution
U1=15, I1=0.5, R1=12, R2=5, R3=8, R4=3, k1=2, k2=1.5
```

```
% find the result of substituting the above values into the symbolic expressions
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))) ); end
 v1:  15.000000
 v2:   6.176471
 v3:  -2.161765
 v4:  10.985294
 v5: -10.985294
 ia:  -0.735294
 ib:  -1.602941
 ix:   0.735294
 uy:  10.985294
```

Supernode and Simplifications approach (fewer equations, more thinking)

There are, of course, many (very many) different ways to form equations to solve for the potentials. The most mechanical sort of method was shown above. What will be tried here is to identify some node potentials in terms of others:

$$v_1 = U_1 \tag{1}$$

$$v_3 = v_5 + R_3 k_2 \frac{U_1 - v_2}{R_1} \tag{2}$$

$$v_4 = v_5(1 - k_1) \tag{3}$$

This means that only 2 KCL equations are now needed. (We had 6 nodes. KCL isn't needed on one of them anyway, since that would just be saying the same thing as summing the other KCLs: we normally choose to ignore the earth node. We have now expressed three other potentials, v_1, v_3, v_4, in terms of the remaining two v_2, v_5 and known quantities. So just two KCL equations, at nodes 2 and 5 should suffice.

$$\text{KCL}(2): \quad 0 \quad = \quad \frac{v_2 - U_1}{R_1} + \frac{v_2}{R_2} + k_2 \frac{U_1 - v_2}{R_1} + \frac{v_2 - v_5(1 - k_1)}{R_4} \tag{4}$$

$$\text{KCL}(5): \quad 0 \quad = \quad -I_1 - k_2 \frac{U_1 - v_2}{R_1} - \frac{v_2 - v_5(1 - k_1)}{R_4} \tag{5}$$

Together, the above set of 5 equations in the 5 unknown node potentials should be able to give our solution.

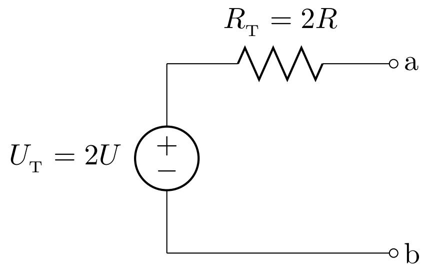
```
s = solve( ...
  'v1 = U1', ...
  'v3 = v5 + R3*k2*(U1-v2)/R1', ...
  'v4 = v5*(1-k1)', ...
  '0 =  (v2-U1)/R1 + v2/R2 + k2*(U1-v2)/R1 + (v2 - v5*(1-k1))/R4', ...
  '0 =  -I1 - k2*(U1-v2)/R1 - (v2 - v5*(1-k1))/R4', ...
  'v1,v2,v3,v4,v5' )

% set values, and substitute into the expressions
U1=15, I1=0.5, R1=12, R2=5, R3=8, R4=3, k1=2, k2=1.5
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))) ); end
 v1:   15.000000
 v2:    6.176471
 v3:   -2.161765
 v4:   10.985294
 v5:  -10.985294
```

That seems to have worked! It was shorter, and it gave us all that we had to find. But we did not get the extra answers, about the currents in the sources. And if the circuit were changed, it could require significant work to change the equations: the 'extended' method keeps a very simple relation between equations and topology.

The above was almost the standard supernode method: it avoided needing to know the currents in voltage sources, which it achieved by using those source values to relate node potentials to each other, thereby reducing the number of nodes where KCL was needed. But we also did this for node 3, by noticing that the voltage across R_3 is determined by the CCCS $k_2 i_x$, where i_x can in turn be expressed in terms of node potentials and known component values. This is just an example. For later exams, the extended method is recommended for this type of question. For further understanding and some quick practical circuits, the simplifications+supernode approach can be useful.

Q3

a) With terminals a-b in the open-circuit condition, all the current I must pass through R_2, and the voltage across R_4 must be zero. The terminal voltage u_{ab} is therefore $U + IR_2$, which is $U + \frac{U}{R}R = 2U$ when expressed in only the known quantities.



The Thevenin resistance at a-b can be found most easily by setting the sources to zero and finding the resistor equivalent (this method is possible because there are not dependent sources in the circuit). Then R_4 and R_2 are in series; the current-source branch is open, and the voltage-source short-circuits R_1. The result is R_2+R_4, which is $2R$ in terms of the known quantities.

It is important to show how the terminals a-b are related to the poles of the voltage source in the Thevenin equivalent. This is conveniently done by a diagram.

b) The maximum power output from the two-terminal circuit (or its equivalent) is obtained when the current and voltage are half their (respectively) short-circuit and open-circuit values. Hence, the maximum power is

$$P = \frac{U_{\text{T}}}{2} \cdot \frac{U_{\text{T}}}{2R_{\text{T}}} = \frac{2U}{2} \cdot \frac{2U}{2 \cdot 2R} = \frac{U^2}{2R}.$$

c) We already stated that the maximum power condition is when the voltage is half the open-circuit value, i.e. $u_{\text{ab}} = \frac{U_{\text{T}}}{2} = U$. This is the value that the external source should have, to extract maximum power from the shown circuit: $U_x = 2U/2$, so $U_x = U$. We should be careful to note that the extra source needs its + side towards the terminal 'a', since the value of U_{T} was derived for 'a' with respect to 'b'. Otherwise, a minus sign would be needed.

Q4 Remember to **redraw** the circuits in your own solution ... almost certainly useful!
Quick answers are given below: check the derivation with your diagrams.

a)

$$i_1(\infty) = 0 \quad \text{(equilibrium)}, \qquad u_3(\infty) = I\frac{R_3 R_4}{R_3 + R_4}.$$

b)

$$i_1(0^+) = 0, \qquad u_3(0^+) = I\frac{(R_1 + R_2 + R_3)R_4}{R_1 + R_2 + R_3 + R_4}, \qquad i_2(0^+) = I\frac{R_1 + R_2}{R_1 + R_2 + R_3 + R_4}.$$

Most of the above are fairly clear if one draws the diagrams carefully, replacing C and L with with open- and short-circuits for the equilibrium conditions, or voltage- and current-sources for the short time at 0+ when continuity can be assumed.

The hardest is $i_2(0^+)$. The first step is to find the voltage on C_2, which can be done from equilibrium and continuity. It comes out as $u_{\text{C2}} = R_3\frac{R_4}{(R_1+R_2+R_3)+R_4}I$, by current division and Ohms' law in the state at $t = 0^-$. When the switch closes, R_3, R_4 and C_2 are all in parallel, and the current in L_3 at $t = 0^+$ is still equal to I by continuity. Ohm's law with $u_{\text{C2}}(0^+)$ gives the current in the two parallel resistors, $R_3\frac{R_4}{(R_1+R_2+R_3)+R_4}I\frac{R_3+R_4}{R_3 R_4}$. The difference between this and the current I in the inductor is i_2, by KCL.
Some rearrangement gives the above answer.

Q5

There is just one change in this circuit over all time: the switch closes at $t = 0$. We need to find the voltage on the inductor for all time after this change. The initial value of the inductor's continuous variable (current) will be important: it can be found by continuity, from the equilibrium before the switch closed.

Before the switch closes, equilibrium can be assumed, since the circuit has been standing "a long time" (i.e. since $t = -\infty$) and the circuit has no special cases such as a voltage source directly across an inductor, or a dependent source that gives a negative resistance. The inductor can therefore be treated as a short-circuit: it has a steady current, so its voltage is zero. The inductor's current (let's define it as i, downwards) can then be found by KCL in the top node, which we can label as v, treating the bottom as zero.

$$0 = \frac{v-(-U)}{R_2} + \frac{v}{R_1} + I$$

from which $v = -\frac{R_1(U+IR_2)}{R_1+R_2}$, so

$$i(0^+) = i(0^-) = -\frac{U+IR_2}{R_1+R_2}.$$

When the switch closes, the branch of L and R_1 is connected to a short-circuit. All the other components thus become irrelevant to what happens around L. KVL around the loop gives that $u(t) + iR_1 = 0$, which can be expressed as

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t)R_1 = 0.$$

We can solve for $i(t)$ then find the requested $u(t)$ from that. The final value, $i(\infty)$ is zero: this can be seen by noticing that current in R_1 loses energy, but there is no source in this branch; or consider that the inductor L is connected to a Thevenin source with zero voltage and resistance R_1.

The current can then be found (by considering final and initial values in a first-order circuit, or by solving the differential equation), as $i(t) = -\frac{U+IR_2}{R_1+R_2}\mathrm{e}^{-tR_1/L}$ for $t > 0$.
The voltage $u(t)$, by $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, is

$$u(t) = -L\frac{-R_1}{L}\cdot\frac{U+IR_2}{R_1+R_2}\cdot\mathrm{e}^{-tR_1/L} = \frac{(U+IR_2)R_1}{R_1+R_2}\,\mathrm{e}^{-tR_1/L} \quad (t>0).$$

Q6

a) Let's translate this time-domain current $I(t) = \hat{I}\sin(\omega t)$ to a phasor $I(\omega) = \hat{I}\underline{/0}$. (This implies we're using peak magnitude and sine-reference. You might have preferred e.g. a cosine reference. Then the final time-domain solution of $i_\mathrm{c}(t)$ should be equivalent to the ones found here, but the phasors will be different during the calculation.)

By current division,

$$i_\mathrm{c}(\omega) = I(\omega)\,\frac{R+\mathrm{j}\omega L}{R+\mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)}.$$

Now this current needs to be converted back into a time-function. First, express it in polar form, to get the magnitude and phase for the time-function. The big question: shall we find these properties for the top and the bottom, then add in polar form, shall we put the whole expression into rectangular form then convert? Let's show both. First, separate conversion of top and bottom to polar form,

$$i_\mathrm{c}(\omega) = I(\omega)\,\frac{R+\mathrm{j}\omega L}{R+\mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)} = \hat{I}\sqrt{\frac{R^2+(\omega L)^2}{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}}\;\underline{/\tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}}.$$

Ok. That didn't look *too* bad, but perhaps we'd like to avoid having two inverse tangents. Let's try the other approach, where we first make the complex part be only on the top,

$$i_c(\omega) = I(\omega)\,\frac{R + j\omega L}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \hat{I}\,\frac{R^2 + \omega^2 L^2 - \frac{L}{C} + j\frac{R}{\omega C}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

and then express this in polar form with a single inverse tangent,

$$i_c(\omega) = \frac{\hat{I}\sqrt{\left(R^2 + \omega^2 L^2 - \frac{L}{C}\right)^2 + \left(\frac{R}{\omega C}\right)^2}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\,\angle \tan^{-1}\frac{\frac{R}{\omega C}}{R^2 + \omega^2 L^2 - \frac{L}{C}}.$$

which unfortunately seems not better to look at than the earlier result. The choice of a better solution would depend on what we want to use the solution for. (In the above expression, it would also be necessary to check the sign of the $R^2 + \frac{L}{C} - \omega^2 L^2$ term, and to add π to the angle if it is negative. We can avoid thinking of this when we do numerical calculations on computers, making use of a suitable function, like `angle()` in Octave/Matlab, to find the phase of a complex number.)

Let's take the earlier form.

$$i_c(t) = |i_c(\omega)|\sin\left(\omega t + \angle i_c(\omega)\right) = \hat{I}\sqrt{\frac{R^2 + (\omega L)^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\ \sin\left(\omega t + \tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}\right).$$

b) By current division, this time finding the current to the left branch, the familiar $|i|^2 R$ formula for power can be used. We must remember that we chose the phasor to be a peak value, so the power calculation must include the factor of $\frac{1}{2}$.

$$P_R = \left|\hat{I}\frac{\frac{1}{j\omega C}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}\right|^2 \frac{R}{2} = \frac{\hat{I}^2}{(\omega C R)^2 + (1 - \omega^2 L C)^2}\,\frac{R}{2}.$$

c) Answer: The same power as in part 'b'! Power superposition can be a useful principle ...
Reason: In the original circuit, the active power supplied by the current source is equal to the active power consumed by the resistor: there's nowhere else for the active power to go. When another source, *with different frequency*, is added to the circuit, power superposition allows the contribution to powers at each frequency to be treated separately (with sources at other frequencies 'set to zero'). If this is done for calculating the power from the current source, the voltage source would be set to zero when the current source is considered: then the circuit is like the original circuit, as L and R are connected by a short-circuit, so the current source's power output is the same as before. Another way of seeing it is that the voltage source *will* contribute a voltage across the current source, but this voltage is at a different frequency from the current, so they make no (average) power together.

Q7

a) The output of the left opamp is labelled v_m. We can see the circuit as two opamp-based inverting amplifiers, where the left one drives the input of the right one. The point v_m is a stiff voltage source, i.e. its "source impedance" is zero: this potential will not be affected by the current that goes into R_3 and C_3 (because the left opamp always adjusts this voltage to the level that forces its inputs to be equal). So we can find the separate network functions of the left and right circuits, then multiply them.

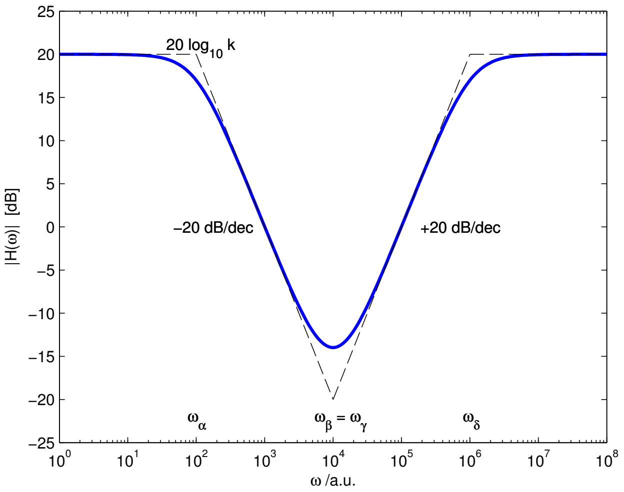
$$H(\omega) = \frac{v_o}{v_i} = \frac{v_m}{v_i}\cdot\frac{v_o}{v_m} = \frac{-\frac{1}{\frac{1}{R_2} + j\omega C_2}}{R_1 + j\omega L_1}\cdot\frac{-(R_4 + j\omega L_4)}{\frac{1}{\frac{1}{R_3} + j\omega C_3}} = \frac{(R_4 + j\omega L_4)\left(\frac{1}{R_3} + j\omega C_3\right)}{\left(\frac{1}{R_2} + j\omega C_2\right)(R_1 + j\omega L_1)}$$

b) By multiplying the four terms in the above expression for $H(\omega)$ by suitable values of $1/R_4$, R_3, etc, then multiplying the whole expression by the reciprocals of these, the more standard network-function form can be found,

$$H(\omega) = k\cdot\frac{(1 + j\omega/\omega_\beta)(1 + j\omega/\omega_\gamma)}{(1 + j\omega/\omega_\alpha)(1 + j\omega/\omega_\delta)}.$$

where $k = \frac{R_2 R_4}{R_1 R_3}$, $\omega_\alpha = \frac{1}{L_1/R_1}$, $\omega_\beta = \frac{1}{C_3 R_3}$, $\omega_\gamma = \frac{1}{L_4/R_4}$, and $\omega_\delta = \frac{1}{C_2 R_2}$.
Note: we could swap $\omega_\alpha \leftrightarrow \omega_\delta$ or $\omega_\beta \leftrightarrow \omega_\gamma$, as there is no numerical data in the question to say which pair of components provides each pole (denominator term) or each zero (numerator term) in the network function.

c) This figure shows the classic asymptotic approximation (dashed lines) which was all that was necessary for this question, along with the actual function $|H(\omega)|$. The choice of frequency is arbitrary (a.u. means 'arbitrary units'), but the ratios must be correct: we know, for example, that $\omega_\delta = 100\omega_\gamma$.



There is no pure ω-term (e.g. $\mathrm{j}\omega/\omega_x$) so the low-frequency value, $\omega \to 0$ is $H(\omega) \to k$. With the given relations, where $\omega_\alpha \ll \omega_\beta = \omega_\gamma \ll \omega_\delta$, the magnitude $|H(\omega)|$ starts decreasing at 20 dB/decade when the frequency exceeds ω_α. Then, at $\omega = \omega_\beta = \omega_\gamma$, the two 'zeros' (terms on the top of the network function) become active, each contributing an *increase* of 20 dB/decade: the total result is an increase of 20 dB/decade. Finally, when $\omega > \omega_\delta$, the other 'pole' (term on the bottom) becomes active, giving a slope that cancels the second zero.

As we were even told the numeric value of k and the numeric relations between the frequencies $\omega_{\alpha,\beta,\gamma,\delta}$, we can use the known changes in dB/decade to find the numeric values of $|H(\omega)|$. There are two decades ($\times 100$) between ω_α and ω_β, and the same factor again to go up to ω_δ: the change is therefore 40 dB for both of the sloping regions. The final value, of $|H(\omega)|$ when $\omega \gg \omega_\delta$, can be found either by starting at the low-frequency value and following the slopes, or algebraically by ignoring all the '1+' terms, $H(\gg \omega_\delta) \simeq \frac{k\omega_\delta\omega_\alpha}{\omega_\beta\omega_\gamma}$. With the given relations of the ω_x values, this becomes simply k.

Q8

a) This is just a form of the familiar ac maximum-power problem, with a Thevenin source and a load impedance. As U, R_1 and L are fixed, it is convenient to see these as the 'source', and the components that we are free to choose can be the 'load'.

It's best to start by equating *admittances*. That's because the two unknowns (R_2 and C) are in parallel, so they independently determine the real and imaginary parts of the admittance. That makes it easy to equate each of these components separately with the real or imaginary part of the total impedance.

Using the ac maximum power criterion of $Z_{\text{load}} = Z_{\text{src}}^*$, but with admittances,

$$Y_{\text{src}}^* = \left(\frac{1}{R_1 + \mathrm{j}\omega L}\right)^* = \frac{R_1 + \mathrm{j}\omega L}{R_1^2 + \omega^2 L^2} = Y_{\text{load}} = \frac{1}{R_2} + \mathrm{j}\omega C.$$

The real and imaginary parts can then be equated directly to find R_2 and C,

$$R_2 = \frac{R_1^2 + \omega^2 L^2}{R_1}, \qquad C = \frac{L}{R_1^2 + \omega^2 L^2}.$$

b) The quickest way to answer this is to find the maximum power possible from the source. That can be found purely from the known quantities describing the source. The advantage is that we then don't have to use the expressions from part 'a' to convert the unknown load components into known quantities.

Consider a load impedance that is chosen for maximum power, $Z_{\text{load}} = Z_{\text{src}}^*$. In the circuit consisting of the voltage-source U, then source impedance $Z_{\text{src}} = R_1 + \mathrm{j}\omega L$, then load $Z_{\text{load}} = R_1 - \mathrm{j}\omega L$, the reactive components cancel. The total impedance in this loop is therefore just $2R_1$, which is the same as the dc maximum power situation if the reactive components are ignored. The power to the load is then $P_{\max} = \frac{U^2}{4R_1}$.
(The 4 is because the current is $U/2R_1$, and the power to the load is $|i|^2 R_1$; there are a few other ways to reason to the same conclusion. If the phasor $U\underline{/0}$ had been a peak value instead of rms, then a *further* $1/2$ would have been needed.)

A longer way to get the solution would be to solve the circuit using all the components, to find the actual power in R_2, and then to substitute the calculated values for R_2 and C from part 'a' and do some simplification. This also comes to the above result of $\frac{U^2}{4R_1}$, but it might not be obvious from the resulting expressions! It's easier to make a simplifying choice from the start, thereby avoiding the extra work.

c) The addition of a transformer makes it look worrying. It's not that bad. The transformer could be seen as part of the source, or part of the load. We know that impedances can be 'translated' across the transformer by the ratio n^2, and voltages can be translated by n (or reciprocals, depending on which side the components are on, and whether impedance or admittance is being considered). In fact, if $n = 1$ in the lower circuit, then it is equivalent to the upper circuit.

It seems easiest to consider the transformer as being part of the load. That way, only the impedances R_2 and $\frac{1}{\mathrm{j}\omega C}$ need to be translated across the transformer.

We find the maximum power condition by making the total load-admittance (seen by the source at the left winding of the transformer) be equal to the conjugate of the source impedance. A simple change is needed in the derivation from part 'a',

$$Y_{\text{src}}^* = \left(\frac{1}{R_1 + \mathrm{j}\omega L}\right)^* = \frac{R_1 + \mathrm{j}\omega L}{R_1^2 + \omega^2 L^2} = Y_{\text{load}} = \frac{1}{n^2}\left(\frac{1}{R_2} + \mathrm{j}\omega C\right).$$

The real and imaginary parts can then be equated directly to find R_2 and C,

$$R_2 = \frac{R_1^2 + \omega^2 L^2}{n^2 R_1}, \quad C = \frac{n^2 L}{R_1^2 + \omega^2 L^2}.$$

Q9

a) Each impedance Z has the line-voltage U across it, as they are in Δ-connection. The complex power into each is U^2/Z^*; remember that U is the magnitude, not a phasor, so it is not necessary to take the absolute value $|U|$. As each element Z is two parallel components, it is convenient to calculate as $U^2 Y^*$, where $Y = \frac{1}{R} - \mathrm{j}\frac{1}{\omega L}$. The total complex power into the Δ load is therefore $S = 3U^2\left(\frac{1}{R} + \mathrm{j}\frac{1}{\omega L}\right)$, from which $P = \frac{3U^2}{R}$ and $Q = \frac{3U^2}{\omega L}$.

b) The only way of getting PF $= 1$ is to ensure that there is no reactive power flow at the terminals. Thus, the component chosen for Z_x must cancel the reactive power of the load Z. This requires a capacitor, as the reactive part of Z is inductive. The solution is quite simple, as Z has parallel R and L, which means that its reactive power consumption depends only on L, not R. The only complication, making the problem less trivial than '$\omega L = \frac{1}{\omega C}$', is that the compensation components Z_x are Y-connected. For equal-and-opposite reactive power,

$$3\frac{U^2}{\omega L} = 3\left(U/\sqrt{3}\right)^2 \omega C = U^2 \omega C,$$

which requires that each Z_x is a capacitor $C = \frac{3}{\omega^2 L}$. Try drawing this as complex numbers representing admittance or current, if that helps you to understand the algebra.

c) If the parallel R and L in each impedance Z are related by $\omega L = R/\sqrt{3}$, then this load has a PF $= {}^1\!/_2$ lagging. Bearing in mind that the R and L are in parallel, it is easiest to think of their admittances. The magnitude and real and imaginary parts of the admittance of load Z are proportional to the apparent, active and reactive powers, due to the parallel connection. From this, PF $= \frac{P}{|S|} = \frac{1}{R} \Big/ \sqrt{\frac{1}{R^2} + \frac{1}{\omega^2 L^2}}$, which after subsitution of $\omega L = R/\sqrt{3}$ becomes ${}^1\!/_2$. We know it is lagging, as the reactive part is an inductance. (To work with power instead of admittance, we would just include a common factor of U^2, or of $3U^2$ for this whole 3-phase load, and multiply by the conjugate of admittance; the result for power-factor is the same.)

In part 'b', an expression was found for a capacitor C that could be used as Z_x to give complete power-factor compensation. We might want to compensate only with the smallest possible current into Z_x that fulfills the requirement of $\geq {}^1\!/\!\sqrt{2}$ lagging: that is probably cheaper than having full compensation, if smaller components can be uesd. If the capacitance C from part 'b' is reduced, the power factor will reduce from 1.0 and be lagging. How *much* capacitance is needed to achieve just PF $= {}^1\!/\!\sqrt{2}$? This power-factor implies that the active and reactive power have the same magnitude. So our aim is that the capacitor for Z_x should result in the total reactive power equalling the total active power. Using the subscripts 'Δ', 'x' and 'total' for the separate Δ and Y loads and their sum,

$$Q_\Delta + Q_x = \frac{3U^2}{\omega L} + (-U^2 \omega C) = Q_{\text{total}},$$

and we want

$$Q_{\text{total}} = P_{\text{total}} = P_\Delta = \frac{3U^2}{R} = \frac{3U^2}{\sqrt{3}\omega L},$$

from which

$$Q_x = P_\Delta - Q_\Delta, \quad \text{so} \quad -U^2 \omega C = \frac{3U^2}{\omega L}\left(\frac{1}{\sqrt{3}} - 1\right),$$

giving $C = \frac{3-\sqrt{3}}{\omega^2 L}$, which could be expressed in a variety of ways. This is less than half of the size of capacitance needed for full power-factor compensation.

When the requirement is only to make the power factor get somewhat higher, we could add conductance (a resistor) for Z_x instead of a capacitor. This cannot compensate the reactive power of Z, but it can increase the total active power and thus the total apparent power. The power factor therefore rises, although it cannot reach exactly 1 as long as the load has some reactive power. That's a silly cheat: normally the main purpose of power factor 'improvement' is to reduce the apparent power, which is related to the current through the wires, and thus to power losses and required materials. The power-factor is the ratio between the active and apparent power, which is usually only worth optimising by reducing the reactive power.

Anyway: let's suppose we *do* want to choose Z_x to be a pure resistance R_x, that gives just the minium required power-factor of ${}^1\!/\!\sqrt{2}$ with the least possible current in Z_x. Then we need to increase the total active power until it equals the reactive power. This can be done by making Z_x be a resistor R_x, with a value of $R_x = \frac{\omega L}{3-\sqrt{3}}$. It could instead be expressed in terms of the resistance R, as $R_x = \frac{R}{\sqrt{3}(3-\sqrt{3})}$.

Check this! At the terminals a,b,c, the input is $P_{\text{total}} = 3U^2 \frac{1}{R} + U^2 \frac{\sqrt{3}(3-\sqrt{3})}{R}$ and $Q_{\text{total}} = 3U^2 \frac{1}{\omega L}$. The active power can be rewritten in terms of $\omega L = R/\sqrt{3}$, as $P_{\text{total}} = 3U^2 \frac{1}{\sqrt{3}\omega L} + U^2 \frac{3-\sqrt{3}}{\omega L}$. Using these active and reactive powers, the power factor is $\frac{P_{\text{total}}}{\sqrt{P_{\text{total}}^2 + Q_{\text{total}}^2}}$ at the terminals, which simplifies to $\frac{1}{\sqrt{2}}$, as requested.