KTH EI1120 Elkretsanalys (CENMI), Omtenta 2015-06-11 kl 14–19

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, …).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_k vara poängen man fick från kontrollskrivning KS1, och b_k poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a,a_\mathrm{k})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b,b_\mathrm{k})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a,a_\mathrm{k})+\max(b,b_\mathrm{k})+c+h}{A+B+C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Examinator: Nathaniel Taylor

Del A. Likström

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1) [4p]

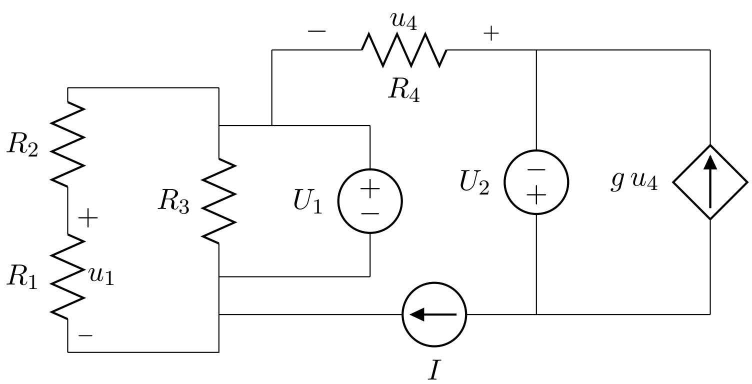
Bestäm de följande:

a) [1p] Effekten levererat till R_3.

b) [1p] Spänningen u_1 över R_1.

c) [1p] Effekten levererat till R_4.

d) [1p] Effekten levererat från källan U_2.

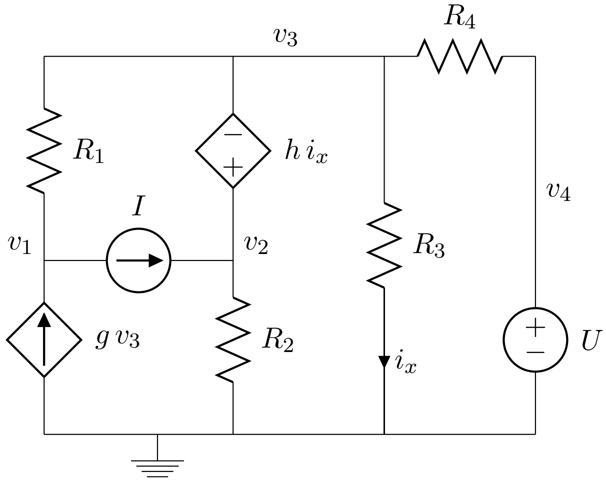


2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1, v_2, v_3, v_4.

Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.

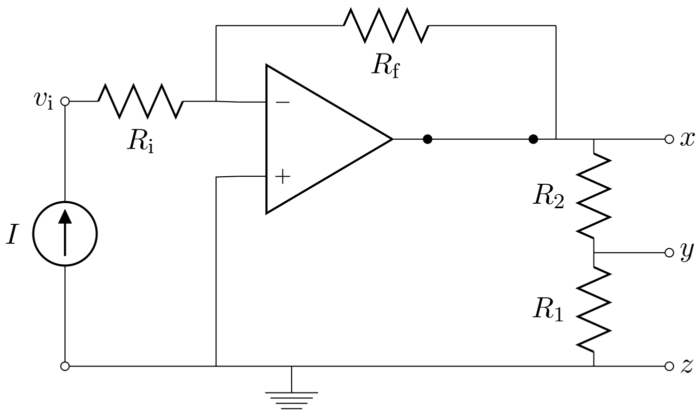
Som vanligt är det komponentvärdena R_1, I, g, h o.s.v. som är kända, medan de markerade storheterna v_1, i_x, o.s.v. är okända.



3) [4p]

a) [1p] Bestäm förstärkarkretsens ingångsresistans, d.v.s. $R_{\text{in}} = v_{\text{i}}/I$.

b) [2p] Vad är Nortonekvivalenten av kretsen, sett mellan polerna y och z? Rita upp Nortonkällan och resistansen för att visa sambandet mellan polerna och källans riktning.

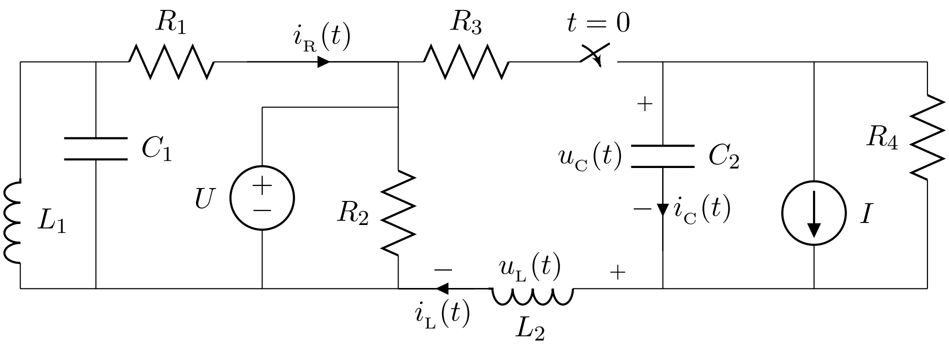


c) [1p] Kortslutningen mellan opamp-utgången och nod x är nu ersatt av ett motstånd R_{o}. Vad är Theveninekvivalenten av kretsen sett mellan polerna x och z? (Obs: inte y och z.)

Del B. Transient

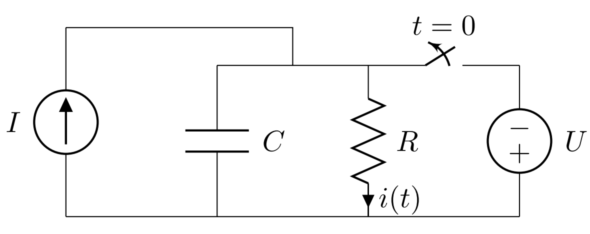
4) [5p]

Bestäm:
$i_{\text{R}}(0^-)$
$u_{\text{C}}(0^-)$
$i_{\text{C}}(0^+)$
$u_{\text{L}}(0^+)$
$i_{\text{L}}(\infty)$.



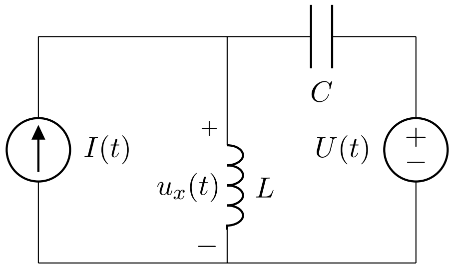
5) [5p]

Bestäm $i(t)$, för $t > 0$.



Del C. Växelström

6) [4p]



Källornas värden är $U(t) = \hat{U}\sin(\omega t)$ och $I(t) = \hat{I}\cos(\omega t)$.

a) [3p] Bestäm $u_x(t)$.

b) [1p] Är det möjligt för källan $I(t)$ att levererar aktiveffekt? Med andra ord, kan positiva värden av $\hat{I}$, $\hat{U}$, L, C, ω väljas som gör att strömkällan leverera aktiveffekt i denna krets? Förklara ditt svar.

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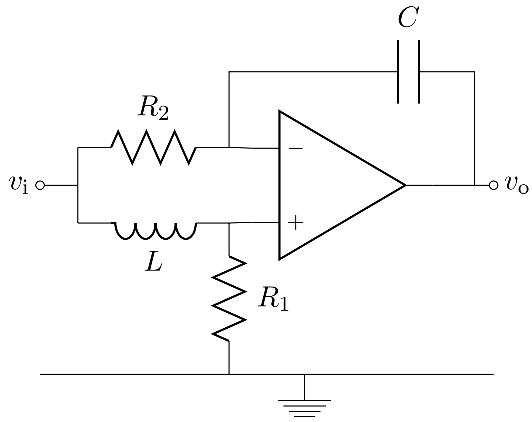
7) [5p]



a) [2p] Bestäm nätverksfunktionen $H(\omega) = \frac{v_\mathrm{o}(\omega)}{v_\mathrm{i}(\omega)}$ av kretsen.

b) [2p] Visa att $H(\omega)$ i deltal 'a' kan skrivas som

$$H(\omega) = \frac{k}{1 + \mathrm{j}\omega/\omega_x},$$

där k och ω_x är positiva reella tal (vid antagandet att komponentvärdena också är det).

c) [1p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'. Markera viktiga punkter och lutningar.

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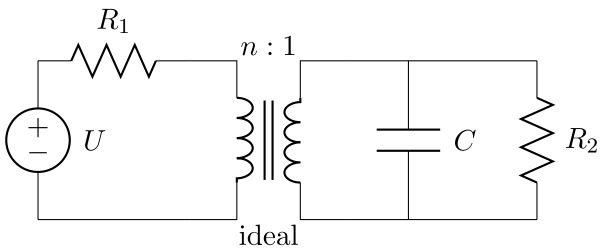
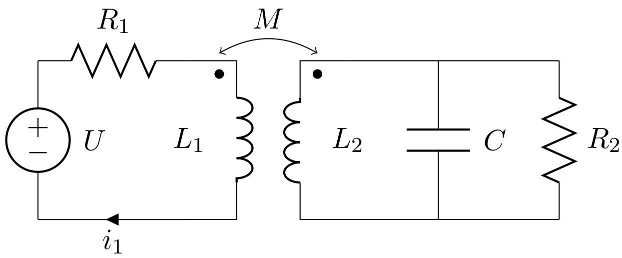
8) [5p]





Källan ger en växelspänning med vinkelfrekvens ω, beskriven av fasvektorn U.

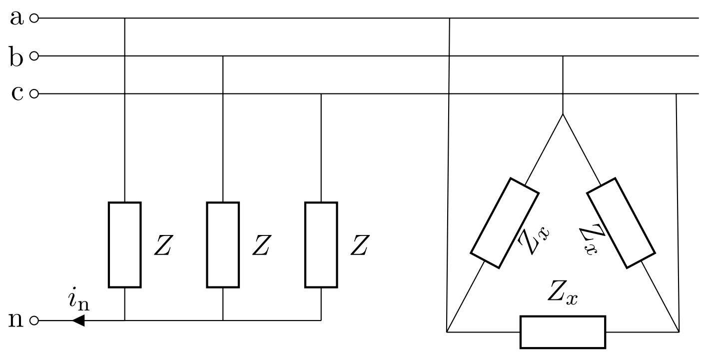
a) [3p] Betrakta den övre kretsen, med en ideal transformator. Komponentvärdena R_2 och C är okända, men de andra komponentvärdena är kända storheter. Välj R_2 och C för att maximera effekten som levereras till motståndet R_2.

b) [2p] Betrakta nu den nedre kretsen, där en ickeideal transformator beskrivs med kopplade spolar. Alla sju komponentvärdena är kända storheter. Bestäm i_1.

9) [4p]

Polerna a,b,c,n visar en anslutning till en balanserad trefas spänningskälla, av vinkelfrekvens ω och huvudspänning magnitud U: $|u_{\text{an}}| = |u_{\text{bn}}| = |u_{\text{cn}}| = \frac{U}{\sqrt{3}}$. Som vanligt när det gäller elkraft, är det ett effektivvärde.

Varje impedans Z representerar ett motstånd R och en spole L, *parallell*-kopplade. Värdet av Z_x ska bestämmas. De kända storheterna är U, ω, R och L.



a) [2p] Vilken aktiv effekt och reaktiv effekt förbrukas av Y-lasten (de tre impedanserna Z)?

b) [1p] Bestäm ett slags komponent (spole, kondensator, eller motstånd) för Z_x, och dess värde, så att effektfaktorn (PF) av alla sex impedanser, sett från källan vid polerna a,b,c,n, blir 1.

c) [1p] Vad är strömmen i_{n} om impedansen mellan fas-b och neutralledaren ändras från Z till $2Z$, medan de andra två impedanserna i Y-lasten fortfarande är Z? Fasföljden är a-b-c, och fasspänningen av fas-a kan tas som referensvinkeln, d.v.s. $\angle u_{\text{an}} = 0$.

Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1120, VT15, 2015-06-11)

Q1

a) Power into R_3 is U_1^2/R_3, as the resistor is parallel with the voltage source U_1, which fixes its voltage and makes all the other components irrelevant to the solution.

b) Voltage u_1 is $U_1 \dfrac{R_1}{R_1+R_2}$, as the series pair R_1 and R_2 is in parallel with the voltage source U_1 which fixes their voltage and makes all other components than these three irrelevant to u_1.

c) Power into R_4 is $I^2 R_4$. This resistor is in series with the independent current source I! It may not be obvious immediately, but it can be seen by considering the whole region of $\{ R_1, R_2, R_3, U_1 \}$: KCL says that the current in from I has to pass out into R_4. Alternatively, consider KCL on the right-hand side of the circuit, for voltage source U_2 and the dependent current source $g\,u_4$.

d) The current through the voltage source U_2 needs to be found, in order to find what power this source delivers. Let's define the source's current as i_2, out from the '+' terminal. Then by KCL at the node below U_2, $i_2 = I + gu_4$. By seeing that the current from right to left in R_4 must be $-I$, we get $u_4 = -IR_4$. Substituting this into the earlier expression gives $i_2 = I\,(1 - gR_4)$. The power out from source U_2 is therefore $U_2 I\,(1 - gR_4)$.

Q2

We'll show the "extended nodal analysis" (which I've sometimes called the simple method, because it has few rules although it generates longer equation systems).

First, KCL at every node except ground. We'll define the currents in the voltage sources as i_α (in U) and i_β (in the dependent voltage source), both being into the '+' terminal.

$$\begin{aligned}
\mathrm{KCL}(1):\quad 0 &= \frac{v_1 - v_3}{R_1} + I - g\,v_3 && (1)\\
\mathrm{KCL}(2):\quad 0 &= \frac{v_2}{R_2} - I + i_\beta && (2)\\
\mathrm{KCL}(3):\quad 0 &= \frac{v_3 - v_1}{R_1} - i_\beta + \frac{v_3}{R_3} + \frac{v_3 - v_4}{R_4} && (3)\\
\mathrm{KCL}(4):\quad 0 &= \frac{v_4 - v_3}{R_4} + i_\alpha. && (4)
\end{aligned}$$

Then, include the relations of node potentials that the voltage sources determine,

$$\begin{aligned}
v_4 &= U && (5)\\
v_2 - v_3 &= h\,i_x. && (6)
\end{aligned}$$

Finally, define controlling variables of dependent sources in terms of already defined quantities. In this case, one source had a node potential as its controlling variable, so it needs no equation (it would be just '$v_3 = v_3$').

$$i_x = \frac{v_3}{R_3} \qquad (7)$$

We will leave other methods, such as the supernode approach, as 'an exercise for the reader'.

Q3

a) The node at the opamp inverting input ('$-$' input terminal) is held by the feedback to the potential of the '+' input, which we can see is fixed at zero potential. Therefore, $I = \frac{v_\mathrm{i} - 0}{R_\mathrm{i}}$, meaning that the input resistance seen by the current source is just $R_\mathrm{in} = R_\mathrm{i}$. An amplifier circuit's input resistance is an important part of its specification: it determines how much current will be taken from whatever circuit supplies the input.

b) Given that the inverting input must have zero potential (see above), $\frac{v_x}{R_\mathrm{f}} = -\frac{v_\mathrm{i}}{R_\mathrm{i}}$, in which $v_\mathrm{i} = IR_\mathrm{i}$. By voltage division, the open-circuit voltage across the specified terminals is $u_{yz(\mathrm{oc})} = \frac{v_x R_1}{R_1+R_2}$. These combine to

$$u_{yz(\mathrm{oc})} = \frac{-IR_\mathrm{f}R_1}{R_1+R_2},$$

which shows that the resistor R_i was not relevant to the output voltage (it is in series with a current source). The short-circuit current is $i_{yz(\mathrm{sc})} = \frac{v_x}{R_2}$, which in terms of known quantities is

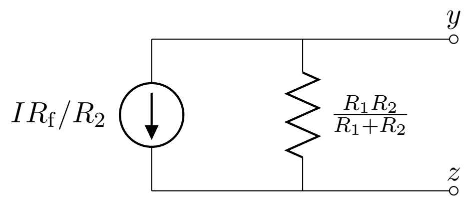
$$i_{yz(\mathrm{sc})} = -IR_\mathrm{f}/R_2.$$

The Norton equivalent has a current source $i_{yz(\mathrm{sc})}$, and a parallel resistance $\frac{u_{yz(\mathrm{oc})}}{i_{yz(\mathrm{sc})}}$. After subsituting the earlier expressions for the short-circuit and open-circuit properties, this resistance can be simplified to

$$\frac{u_{yz(\mathrm{oc})}}{i_{yz(\mathrm{sc})}} = \frac{R_1R_2}{R_1+R_2},$$

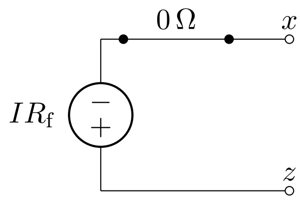
which is simply the parallel resistance of R_1 and R_2. This result could have been expected by realising that the opamp's output has to adjust until the feedback requirement is satisfied: the voltage between x and ground behaves as an ideal voltage source, even though node x is not directly connected to the opamp's output.

The diagram below shows the expected format of the final answer, where it is important to show not only the values of the components but also the way the components are connected (parallel) and the direction of the source with respect to the marked terminals. *Note: an alternative is to draw the diagram with the source in the opposite direction, and a negative sign in the expression for the source's value.*



Left: solution of 'b)'.

Right: solution of 'c)'.



c) It is interesting to note that adding the further resistance R_o in series with the opamp output makes no difference to the potentials or equivalent-source resistance seen at the terminals x and y. The reason is described in part 'b)': the opamp feedback depends on the potential at point x, which gets automatically adjusted to give the necessary feedback current in R_f so that the opamp's inverting input follows the non-interting input. If a larger R_o is added, the opamp just has to give an even greater voltage at its output in order to obtain the necessary v_x to force the inverting input to match the non-inverting input.

Between the point x and ground (z) there is therefore a voltage that is independent of how much current we move between these terminals. This corresponds to a Thevenin voltage v_x and a Thevenin resistance of zero. In this case the method of finding an equivalent source by calculating short-circuit current will not work easily: short-circuiting of x-z destroys the feedback, so the assumption that the opamp inputs have the same potential is no longer true. Examples of methods that can be used instead are: write an equation for the relation of i_{xz} and u_{xz}, that can be arranged in the form $U_\mathrm{T} - R_\mathrm{T} i_{xz} = u_{xz}$; or reason by the feedback argument given above.

Q4

For this sort of calculation it is very helpful to re-draw the circuit at different time-points, omitting irrelevant parts and replacing components with simplifications (such as a short-circuit for an inductor in steady state, or a voltage source for a capacitor immediately after a known equilibrium). The following solutions contain short descriptions of how the solution could be found. It is sometimes very hard to see the solution in this way until systematically re-drawing.

$i_{\mathrm{R}}(0^-) = -U/R_1$ KVL around U, R_1, L_1, treating L_1 as short-circuit.

$u_{\mathrm{C}}(0^-) = -IR_4$ Treat C_2 as open-circuit: then all I passes up R_4.

$i_{\mathrm{C}}(0^+) = 0$ Continuity of current in L_2 means there is still no current through the switch at $t = 0^+$. Continuity of voltage in C_2 means there is the same voltage u_{c} as at $t = 0^-$, which ensures that the current flowing up in R_4 is still I. By KCL above C_2,I,R_4 the current in C_2 is therefore still zero at $t = 0^+$ (note that this current can change when the current through L_2 has had time to change).

$u_{\mathrm{L}}(0^+) = U + IR_4$ Because of continuity of current in L_2, the 'sides' of the circuit (to the left and the right of L_2) see no change between $t = 0^-$ and $t = 0^+$: it takes time before the inductor current can change. A good choice of loop for KVL is L_2, U, R_3, switch, C_2. In this, U is a fixed value, $u_{\mathrm{C}}(0^+) = u_{\mathrm{C}}(0^-)$ by continuity, R_3 has zero current at $t = 0^+$ (see previous solution) and therefore zero voltage, and the closed switch has zero voltage (short-circuit). Hence, KVL gives $-u_{\mathrm{L}}(0^+)+U+0+0-u_{\mathrm{C}}(0^+) = 0$. Comparing the times 0^- and 0^+ we see the voltage that was across the switch when it was open has now appeared across L_2 when the switch closed.

$i_{\mathrm{L}}(\infty) = \frac{U+IR_4}{R_3+R_4}$ After making substitutions for equilibrium (e.g. open circuit for C_2) and removing irrelevant components in parallel with source U (R_2 and all to the left), a single KCL equation $\frac{u_{\mathrm{C}}(\infty)-U}{R_3} + I + \frac{u_{\mathrm{C}}(\infty)}{R_4} = 0$, is sufficient to solve for the voltage $u_{\mathrm{C}}(\infty)$ across the components at the right, and thus for $i_{\mathrm{L}}(\infty) = \frac{U-u_{\mathrm{C}}(\infty)}{R_3}$.

Q5

After the switch opens, the current I of the current source must split between the capacitor and resistor. Let's define the voltage across the capacitor (with the positive reference upwards) as u_{c}, and the current in the capacitor (downwards) as i_{c}. The resistor and capacitor are in parallel, so

$$i(t) = \frac{u_{\mathrm{c}}(t)}{R}.$$

The current in the capacitor is given by KCL as

$$i_{\mathrm{c}}(t) = I - \frac{u_{\mathrm{c}}(t)}{R}.$$

The constitutive equation for a capacitor is $i_{\mathrm{c}}(t) = C\frac{\mathrm{d}u_{\mathrm{c}}(t)}{\mathrm{d}t}$, which allows a differential equation in u_{c} to be written as

$$C\frac{\mathrm{d}u_{\mathrm{c}}(t)}{\mathrm{d}t} = I - \frac{u_{\mathrm{c}}(t)}{R}.$$

It would also have been possible to write a differential equation in another variable such as i_{c} or i (which is the variable we're ultimately solving for); we have chosen to use the continuous variable, since this makes it easy to handle the initial condition.

The initial condition has to be found from the equilibrium before the switch opened. With the switch closed ($t < 0$) the voltage source was connected in parallel with the resistor and capacitor. The capacitor voltage must therefore have been $u_{\mathrm{c}}(0^-) = -U$, by KVL. By continuity, $u_{\mathrm{c}}(0^+) = -U$.

The above differential equation and initial condition can be solved for $u_{\mathrm{c}}(t)$ during $t > 0$. The alternative method is to use the initial value ($u_{\mathrm{c}}(0^+) = -U$), the final value ($u_{\mathrm{c}}(\infty) = IR$), and the time-constant given by the capacitor and the equivalent source that it sees ($\tau = RC$ here) to write the solution directly. Either way,

$$u_{\mathrm{c}}(t) = IR - (U + IR)\,\mathrm{e}^{-\frac{t}{CR}}.$$

We were actually looking for $i(t)$, the current through the resistor. This has already been seen to be $i(t) = u_{\mathrm{c}}(t)/R$, so we can write

$$i(t) = I - \left(\frac{U}{R} + I\right)\mathrm{e}^{-\frac{t}{CR}}.$$

Q6

a) Let's take a sine reference, so that $U(t) = \hat{U}\sin(\omega t)$ becomes $U(\omega) = \hat{U}\underline{/0}$.
Then $I(t) = \hat{I}\cos(\omega t)$ becomes $I(\omega) = \hat{I}\underline{/\pi/2} = \mathrm{j}\hat{I}$.

The phasor equation for KCL in the node above L, omitting for neatness the '(ω)' after the phasors, is

$$-I + \frac{u_x}{\mathrm{j}\omega L} + \mathrm{j}\omega C\,(u_x - U) = 0,$$

from which the sought voltage is

$$u_x(\omega) = \frac{I + \mathrm{j}\omega C U}{\mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)} = \frac{\hat{I} + \omega C\hat{U}}{\omega C - \frac{1}{\omega L}}.$$

This expression has no imaginary parts. Its phase-angle is either 0 or π, depending on the sign of the denominator. (If a different reference, such as $\cos(\omega t)$ or $\sin(\omega t + \pi/2)$ had been used, then the phasor $u_x(\omega)$ would not have zero angle, but it would still have the special feature of being a particular angle or that angle plus π.)

One way of writing the time-function is

$$u_x(t) = \frac{\hat{I} + \omega C\hat{U}}{\omega C - \frac{1}{\omega L}}\,\sin(\omega t)\,.$$

This looks rather different from what we often see when solving ac circuits and writing their time functions. Normally we have phase-angles that can vary over a range; then we have written time-functions in the form $f(t) = A\sin(\omega t + \phi)$ (or with cos instead), where A is positive real. Some square-roots are usually needed for finding the magnitude A, and perhaps an inverse tangent for finding ϕ. In the case above, we have instead allowed the term before the sin function to be positive or negative, to make the phase-angle change by π when the difference between ωC and $1/\omega L$ changes sign. The phasor was already purely real, so the square-roots and inverse tangents were not needed. This choice seemed the simplest way to get the right behaviour. Note that when $\omega C = 1/\omega L$ there is a resonance at which u_x has unbounded magnitude.

b) The complex power *out* of source I is given by $u_x(\omega)I(\omega)^*$, which can be written $-j\hat{I}u_x(\omega)$. Substituting $u_x(\omega)$ found in part 'a', we see that the result is a purely imaginary complex power, for any real values of the variables. Thus, there is no active power delivered by the current source.

However ... it is not sufficient to notice that the inductor and capacitor cannot absorb active power. This is a true point, but it doesn't prove what is asked: there is a further component (the voltage source) in the circuit. It would in general be possible for a voltage source in a circuit to absorb active power from a current source. Whether the active power is in or out or zero, depends on the sources' phase angles and possibly on other components in the circuit. In this particular case the specified phase-angles of the sources, and the chosen components between them, prevent transfer of active power. You can try changing the voltage source to $\hat{U}\cos(\omega t)$ for an example where the current source *can* supply active power (or absorb it, depending on the sign of the denominator term).

Q7

a) The potentials of the two opamp inputs are expected to be the same, due to the negative feedback. Let this potential be v. Then, from KCL separately at the two inputs, we get two equations,

$$\frac{v - v_\mathrm{i}}{\mathrm{j}\omega L} + \frac{v}{R_1} = 0, \qquad \text{and} \qquad \frac{v - v_\mathrm{i}}{R_2} + (v - v_\mathrm{o})\,\mathrm{j}\omega C = 0.$$

Eliminating v and rearranging,

$$\frac{v_\mathrm{o}}{v_\mathrm{i}} = H(\omega) = \frac{(1 + \mathrm{j}\omega C R_2) - (1 + \mathrm{j}\omega L/R_1)}{\mathrm{j}\omega C R_2\,(1 + \mathrm{j}\omega L/R_1)}.$$

This can be further simplified as shown in part 'b)'.

b) Simplification of the result from part 'a)' gives,

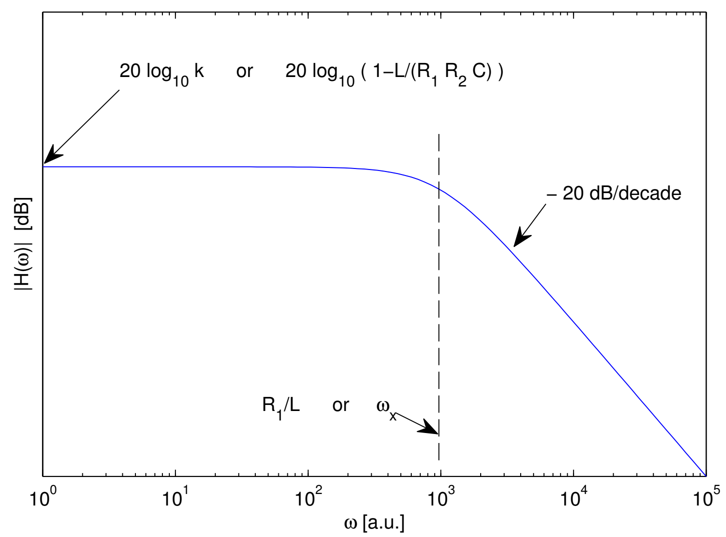
$$H(\omega) = \frac{1 - \frac{L}{CR_1R_2}}{1 + \mathrm{j}\omega L/R_1},$$

which can be written in the requested form of $\frac{k}{1+\mathrm{j}\omega/\omega_x}$ by setting

$$k = 1 - \frac{L}{CR_1R_2} \qquad \text{and} \qquad \omega_x = \frac{R_1}{L}.$$

This is a quite surprisingly simple function: swapping of the positions of the same components in this circuit can result in more complicated Bode plots.

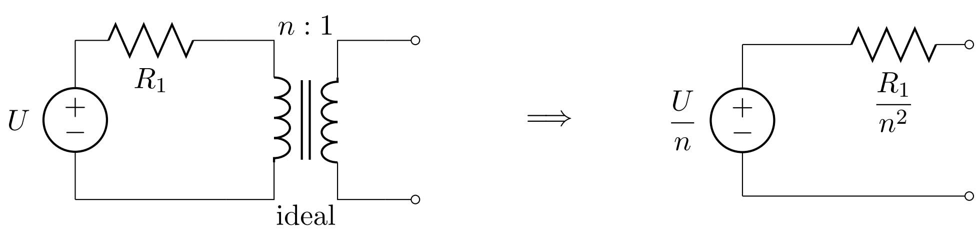
c)



This is an unusually easy Bode amplitude plot. It is just the curve of a single pole, shifted vertically by k (in dB). The values k and ω_x could be used, or their equivalent in terms of circuit quantities such as R_1/L. In case you wonder: 'a.u.' means 'arbitrary units'. Normally the hand-written Bode plot would use straight lines instead of showing the curve around ω_x.

Q8

a) As C and R_2 are able to be chosen, while other components are fixed, it is convenient to see this maximum power question as having a source of U and R_1, a load of C and R_2, and a transformer between. Probably the simplest way to start is to replace the source and transformer by an equivalent source that represents what the load 'sees' at the terminals of the right-hand side of the transformer. If we disconnect the load from the transformer, the open-circuit voltage of the transformer's right-hand coil is U/n. The resistance R_1 is equivalent to a resistance R_1/n^2 on the other side of the transformer. Thus, the following substitution can be made:



Maximum power to the load is attained when the load impedance is the complex conjugate of the source impedance. In this case, the source impedance is pure resistance, so we need $R_2 = R_1/n^2$ and $C = 0$. As usual with maximum power calculations, the source voltage was not relevant to what load impedance should be chosen, although it would affect how large the maximum power is.

b) Let's define u_1 as the voltage on the left coil (L_1), with its reference ('+'-terminal) up; and u_2 on L_2, also with reference up. The current i_1 is already defined; we can define i_2 into the upper end of L_2. Our choice of definitions affects our intermediate equations, but should of course not affect the final answer. (A little note: in this case, with passive components on one side, the dots aren't actually important to the final answer.)

Then, by KVL and the equations for mutual inductors,

$$\begin{array}{rrcl} \text{(KVL, left)} & U & = & i_1 R_1 + \mathrm{j}\omega L_1 i_1 + \mathrm{j}\omega M i_2 \\ \text{(KVL, right)} & u_2 & = & \mathrm{j}\omega L_2 i_2 + \mathrm{j}\omega M i_1 \\ \text{(load)} & i_2 & = & -u_2 \left(\dfrac{1}{R_2} + \mathrm{j}\omega C \right) \end{array}$$

Substituting to eliminate u_2 and i_2,

$$i_1 \;=\; \frac{U}{R_1 + \mathrm{j}\omega L_1 + \mathrm{j}\omega M \frac{-\mathrm{j}\omega M}{\frac{1}{\frac{1}{R_2}+\mathrm{j}\omega C} + \mathrm{j}\omega L_2}} \;=\; \frac{U}{R_1 + \mathrm{j}\omega L_1 + \frac{\omega^2 M^2 (1+\mathrm{j}\omega C R_2)}{R_2 - \omega^2 C L_2 R_2 + \mathrm{j}\omega L_2}}.$$

Q9

a) We're told that the Y-connected load consists of impedances Z, each formed from a parallel resistor R and inductor L. Thus, $\frac{1}{Z} = \frac{1}{R} + \frac{1}{\mathrm{j}\omega L}$. The complex power into this load will be

$$S = 3 \frac{\left(\frac{U}{\sqrt{3}}\right)^2}{Z^*} = U^2 \left(\frac{1}{R} + \mathrm{j}\frac{1}{\omega L} \right),$$

from which the active and reactive parts are $P = \Re\{S\} = U^2/R$ and $Q = \Im\{S\} = U^2/(\omega L)$.

b) The load in part 'a)' consumes reactive power (i.e. it has positive reactive power input), as expected for an inductive load. In order to obtain PF$= 1$, we need to choose Z_x to be a capacitive load, in order to compensate the inductive part of the load Z. This compensation can (equivalently) be described as 'generating' a reactive power that supplies the inductive load, or consuming a negative reactive power that cancels the positive consumption of the inductive load, or generating a cancelling current, or being a parallel resonance of the inductors in Z and the equivalent-Y of the capacitors in Z_x.

Since both of the three-phase loads are balanced, it's easy to write the total reactive power of the two loads. By equating this total to zero (perfect compensation) we can get an equation to solve for what capacitance is needed. Let's call the required capacitance C, so that $Z_x = {}^1\!/_{\mathrm{j}\omega C}$. Then, by the condition that $Q_{\mathsf{Y}} + Q_{\Delta} = 0$,

$$\frac{U^2}{\mathrm{j}\omega L} + 3\frac{U^2}{Z_x^*} \;=\; \mathrm{j}\frac{U^2}{\omega L} - 3U^2 \mathrm{j}\omega C \;=\; 0,$$

from which $C = \frac{1}{3\omega^2 L}$.

By working with powers, and taking advantage of balanced conditions, we avoiding nasty details such as the phasor summation of currents in the delta-connection, the phasor summation between the currents of the two loads, etc.

c) This is now an *unbalanced* load: one phase of the load (phase-b) has had its impedance doubled. That means it's certainly not as simple as the balanced load, where the currents in the star-point (neutral) could be assumed to cancel as long as the source was balanced too. On the other hand, it's not *too* bad — the Y-load does have a neutral connection, so we know that the voltages across the three phase-impedances (Z, $2Z$ and Z) are still balanced three-phase voltages: that's easier than if there had been no neutral connection to the Y load, in which case the unbalanced load would cause the star-point to have a non-zero potential which we'd have to find by KCL.

The method that's simplest to *devise* for this problem (although not necessarily the simplest to solve) is probably just to sum the three currents:

$$i_{\mathrm{n}} = \frac{u_{\mathrm{an}}}{Z} + \frac{u_{\mathrm{bn}}}{2Z} + \frac{u_{\mathrm{cn}}}{Z}.$$

Inserting the given phase-angles of voltages, and doing some simplification based on summing phasors (trigonometry that's not shown here) this is

$$i_{\mathrm{n}} = \frac{U}{\sqrt{3}} \cdot \frac{1}{2Z} \left(2\mathrm{e}^{0} + \mathrm{e}^{-\mathrm{j}\pi 2/3} + 2\mathrm{e}^{-\mathrm{j}\pi 4/3}\right) = -\frac{U}{2\sqrt{3}Z}\mathrm{e}^{-\mathrm{j}\pi 2/3} = \frac{U}{2\sqrt{3}Z}\mathrm{e}^{\mathrm{j}\pi 1/3},$$

or, written in the notation we've more often used,

$$i_{\mathrm{n}} = \frac{U}{\sqrt{3}} \cdot \frac{1}{2Z} \left(2\underline{/0} + 1\underline{/-2\pi/3} + 2\underline{/-4\pi/3}\right) = -\frac{U}{2\sqrt{3}Z}\underline{/-2\pi/3} = \frac{U}{2\sqrt{3}Z}\underline{/\pi/3}.$$

Expressing Z in terms of known quantities R, L and ω (see part 'a'), we find that $|Z| = \frac{\omega LR}{\sqrt{R^2+\omega^2L^2}}$ and $\underline{/Z} = \tan^{-1}\frac{R}{\omega L}$, from which

$$i_{\mathrm{n}} = \frac{U\sqrt{R^2+\omega^2L^2}}{2\sqrt{3}\omega LR}\;\underline{/\pi/3 - \tan^{-1}\tfrac{R}{\omega L}}.$$

All the above fooling around was the algebraic way of just sketching a phasor diagram and saying "the neutral current will have magnitude half as much as the current in the phases in the balanced load Z, and an angle which is opposite the b-phase current in the balanced load Z".

One way to simplify the problem before the equations is to think of the load when *all* impedances are $2Z$ ... then it's balanced ($i_{\mathrm{n}} = 0$), but has half the normal current in each phase; then add a further $2Z$ in parallel, in phase-a and phase-c, to make these each Z: the sum of these new ones is what flows in the neutral, and we see it must be the opposite (180° shifted) of the current that would flow in phase-b for a $2Z$ impedance, since adding a $2Z$ impedance in phase-b would make it a balanced load. How's that for twisted reasoning?

Since we appear to have a little space left on the page, let's see how much less effort it is to handle the complex numbers on a computer, *if the values are known.* We'll assume some numeric values of the known quantities.

```
U = 400,  R = 13.225,  L = 87e-3,  w = 2*pi*50
a = exp(-1j*2*pi/3);  % complex number to shift by -120degrees
uan = U/sqrt(3),  ubn = a*uan,   ucn = a*ubn
Z = 1 / ( 1/R + 1/(1j*w*L) )

% check that the balanced case really does have ~0 neutral-current
in_bal = uan/Z + ubn/Z + ucn/Z

% now solve the unbalanced case, and display in magnitude and degrees
in_unbal = uan/Z + ubn/(2*Z) + ucn/Z
fprintf('\n** neutral current: magnitude %f A, phase %f deg\n', ...
       abs(in_unbal),  angle(in_unbal)*180/pi );
```

The numbers were chosen so that the total 3-phase load would have nice round numbers for its total active power and power factor, off a typical European low-voltage supply. The result for the balanced case was a very small number (`in_bal` was of the order 10^{-15}) meaning that it is zero as expected (apart from numerical inaccuracy). The result for the unbalanced case was 9.7 A at an angle of 34.2°; unless we further calculate the shift between voltage and current due to the load power-factor, or compare this current with the phase-b current, we wouldn't notice that this phasor of the neutral current is exactly opposite the phase-b current. Sometimes it might be useful to understand more of 'what is happening', for which the diagrams and equations can be more handy than the quick numerical approach.