# EI1110 Elkretsanalys (Elektro) Tentamen TEN1, 2015-10-29 kl 14–19

**Hjälpmedel:** Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt.

Tentan har 5 tal i två sektioner: 3 i sektion A (12p), och 2 i sektion B (10p).

Godkänd tentamen TEN1 kräver:

$$\frac{\max(a, a_{\rm k})}{A} \geq 40\% \qquad \& \qquad \frac{b}{B} \geq 40\% \qquad \& \qquad \frac{\max(a, a_{\rm k}) + b}{A+B} \geq 50\%$$

där $A$=12 och $B$=10 är de maximala möjliga poängen från sektionerna A och B, $a$ och $b$ är poängen man fick i dessa respektive sektioner i tentan, och $a_{\rm k}$ är poängen man fick från kontrollskrivning KS1 vilken motsvarar tentans sektion A; funktionen max() tar den högre av sina argument.

Betyget räknas från summan över båda sektioner, igen med bästa av sektion A och KS1, $\frac{\max(a,a_{\rm k})+b}{A+B}$. Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

I vissa gränsfall där betyget är lite under 50%, eller bara en av sektionerna är underkänd trots 50% eller bättre totalt, kommer betyget 'Fx' registreras, med möjlighet att få betyget E om ett kompletterings-arbete är godkänt inom några veckor efter tentamen.

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. $R$ för ett motstånd, $U$ för en spänningskälla, $k$ för en beroende källa) antas vara **kända** storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller en spänningskälla) antas vara **okända** storheter.

Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner.

**Dela tiden** mellan talen — senare deltal brukar vara svårare att tjäna poäng på ... fastna inte!

**Kontrollera** svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

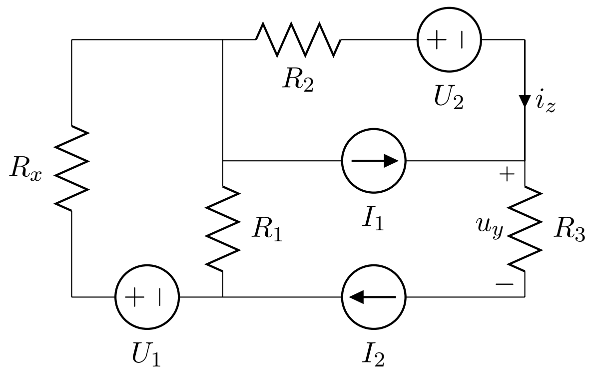
*Använd kvarstående tid* för att kontrollera svaren. Lycka till! Nathaniel Taylor (073 919 5883)

## Sektion A. Likström

**1)** [4p]

a) [3p] Bestäm de följande:
De markerade $u_y$ och $i_z$.
Effekten som källan $U_2$ absorberar.
Effekten som motståndet $R_1$ absorberar.

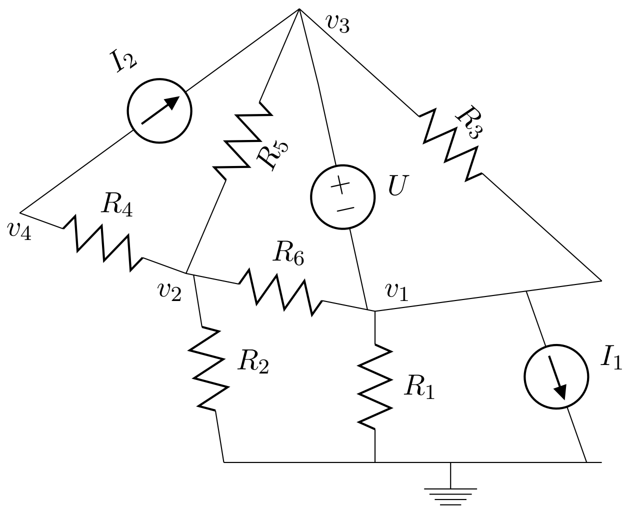
b) [1p] Uttryck $R_x$ som funktion av andra komponent-värden, sådant att den största möjliga effekten levereras till motståndet $R_x$ från resten av kretsen.



**2)** [4p]

Skriv ekvationer som skulle kunna lösas för att finna de markerade potentialerna $v_1$, $v_2$, $v_3$, $v_4$ som funktioner av de givna komponentvärdena.

Du **måste inte** lösa eller förenkla dina ekvationer.
Använd helst en systematisk metod, för att försäkra tillräckliga ekvationer utan onödigt arbete.

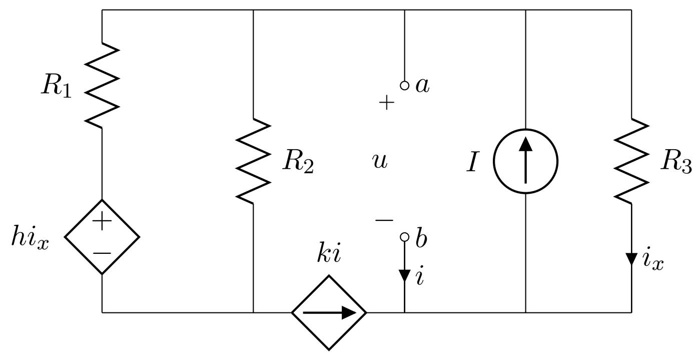



**3)** [4p]

Bestäm Nortonekvivalanten med avseende på polerna $a$ och $b$.

Kom ihåg konventionen att de markerade $u$, $i$ och $i_x$ är okända.

*Ledning:* Vilka komponenter är relevanta för lösningen?




## Sektion B. Transient

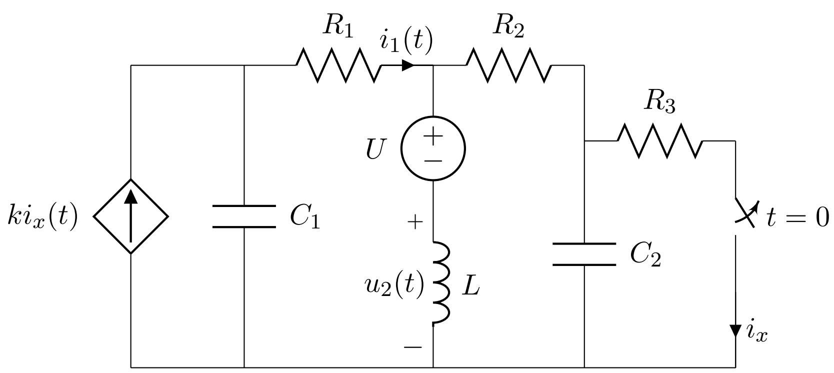
**4)** [5p]

Bestäm $i_1(t)$ och $u_2(t)$ vid de följande tiderna:

a) [2p] $t = 0^-$

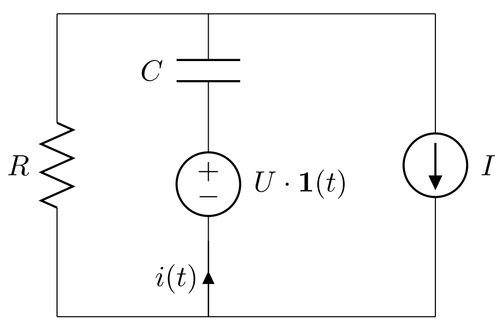
b) [2p] $t = 0^+$

c) [1p] $t \to \infty$




**5)** [5p]

Bestäm $i(t)$ för tider $t > 0$.



# Solutions, EI1110 TEN1 2015-10-29

**1)**

**a)**

$u_y = I_2 R_3$ series with current-source (i.e. KCL)

$i_z = I_2 - I_1$ KCL

$P_{\text{U2}} = i_z U_2 = U_2(I_2 - I_1)$ note: this expression defines the power *into* source $U_2$. For some choices of component values the resulting number will be negative, indicating that in fact this source is supplying power to the rest of the circuit, and for other choices of component values the resulting number will be positive, indicating that the source is absorbing (receiving) power from the rest of the circuit. A source can absorb or produce power, whereas a resistor of positive value can only absorb it (the current is always going from a higher to lower potential, since this potential-change is what drives the current through the resistor).

$P_{\text{R1}} = \left(\frac{U_1 - I_2 R_x}{R_x + R_1}\right)^2 R_1$ this can be found by nodal analysis on one node (i.e. KCL), which allows the voltage across, or current through, $R_1$ to be found. Looking at the node above (or below) $R_1$, there are three branches: $U_1$ and $R_x$ in one branch, $R_1$ is another, and all the other components are equivalent to a source $I_2$. Hence, with $u$ being the voltage across $R_1$, KCL gives $\frac{u-U_1}{R_x} + \frac{u}{R_1} + I_2 = 0$. We could solve for $u$, then find $P_{\text{R1}} = u^2/R_1$. Or substitute $u = iR_1$ where $i$ is the current in $R_1$, and thus solve KCL directly for the current, followed by $P_{\text{R1}} = i^2 R_1$.

**b)** If $R_x$ is to be adjusted to receive the highest possible power from the rest of the circuit, it should have the same value as the Thevenin resistance of the rest of the circuit to which it's connected.

As there are no dependent sources, our easiest way to find the Thevenin resistance (between the nodes where $R_x$ connects, without $R_x$ present) is to set the sources to zero. The part to the right then disappears, due to the series current-sources (open-circuits). The Thevenin resistance is just $R_1$.

$R_{x(\text{maxP})} = R_1$.

**2)** Two solution methods are shown, and a numerical check is made.

**Extended nodal analysis ("the simple way")**

Define the unknown current in voltage source $U$: we'll call it $i_\alpha$, going into the source's + terminal.

KCL (outgoing currents) at all nodes except ground:

$$\text{KCL}(1): \quad 0 = I_1 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_6} - i_\alpha + \frac{v_1 - v_3}{R_3} \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + \frac{v_2 - v_1}{R_6} + \frac{v_2 - v_4}{R_4} + \frac{v_2 - v_3}{R_5} \tag{2}$$

$$\text{KCL}(3): \quad 0 = -I_2 + \frac{v_3 - v_2}{R_5} + i_\alpha + \frac{v_3 - v_1}{R_3} \tag{3}$$

$$\text{KCL}(4): \quad 0 = \frac{v_4 - v_2}{R_4} + I_2. \tag{4}$$

Next, include the further information given by the voltage source,

$$v_3 - v_1 = U. \tag{5}$$

No marked controlling variable needs to be defined (there is no dependent source in this circuit).

**So that's it. The above equations are a sufficient answer.**

### Alternative: Supernode method

KCL is done at each node (or supernode group) apart from the ground node.

There's only one voltage-source, $U$. It connects between nodes 1 and 3, which therefore form a supernode. Only one of the nodes in the supernode needs its potential to be defined: let's use the potential at node 1, $v_1$, in which case the potential in node 3 is $v_1 + U$.

No further substitutions are needed for dependent sources and their controlling variables, as there aren't any in this circuit.

The KCL equations at the supernode and other nodes (other than earth) are

$$\begin{aligned}
\text{KCL(1,3)}: \quad 0 &= I_1 - I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_6} + \frac{v_1 + U - v_2}{R_5} && (1)\\
\text{KCL(2)}: \quad 0 &= \frac{v_2}{R_2} + \frac{v_2 - v_1}{R_6} + \frac{v_2 - v_4}{R_4} + \frac{v_2 - v_1 - U}{R_5} && (2)\\
\text{KCL(4)}: \quad 0 &= \frac{v_4 - v_2}{R_4} + I_2. && (3)
\end{aligned}$$

It is not sufficient to answer with just the above equations, without also saying how to find the remaining potential (even if it's obvious),

$$v_3 = v_1 + U \tag{4}$$

**The above equations are a sufficient answer.**

### Alternative: supernode and simplification

Node 4 just connects two components (a 'trivial node'). One of these is a current source, which determines the current in $R_4$. We can therefore write KCL at $v_2$ using the known $I_2$ instead of the expression $\frac{v_2 - v_4}{R_4}$ (this is like substituting the KCL(4) equation into the KCL(2) equation). We can then write $v_4 = v_2 - I_2 R_4$ to find $v_4$ after solving for the other two potentials.

(Of course, just like the whole supernode approach, this isn't something radically different: we're just finding a way to write the equations in an already-simplified form, instead of simplifying the bigger system of equations from extended nodal analysis.)

$$\begin{aligned}
\text{KCL(1,3)}: \quad 0 &= I_1 - I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_6} + \frac{v_1 + U - v_2}{R_5} && (1)\\
\text{KCL(2)}: \quad 0 &= \frac{v_2}{R_2} + \frac{v_2 - v_1}{R_6} + I_2 + \frac{v_2 - v_1 - U}{R_5} && (2)
\end{aligned}$$

The two other potentials can then be defined, for finding after solving for the above two unknowns.

$$\begin{aligned}
v_3 &= v_1 + U && (3)\\
v_4 &= v_2 - I_2 R_4 && (4)
\end{aligned}$$

The advantage of writing 4 equations this way, instead of the earlier way, is that only two (instead of three) have to be solved simultaneously.

**3)**

$I_{\text{N}} = \frac{I}{1+k}$ and $R_{\text{N}} = (1+k)R_3$.

The Norton source should be drawn, showing the current-source pointing towards the 'a' terminal (or else $I_{\text{N}}$ should be a negative expression) and showing the resistance in parallel with the source.

**One method.** From the terminals a-b, everything to the left is equivalent to the dependent current-source $ki$ (it's a branch including a series current-source). So KCL at the top node gives $ki + i - I + \frac{u}{R_3} = 0$.

At its terminals a Norton source has the $u$-$i$ relation $i = I_{\text{N}} - \frac{u}{R_{\text{N}}}$. By rearranging the KCL to group the terms in a similar way, we can compare the $i$-$u$ relation of the circuit to the $i$-$u$ relation of a generic Norton source,

$$i = \frac{I}{1+k} - \frac{1}{(1+k)R_3}\,u = I_{\text{N}} - \frac{u}{R_{\text{N}}}.$$

from which the Norton parameters are identified.

**Another method.** There are dependent sources in the circuit, so we *cannot* rely on the method of finding the circuit's equivalent resistance by setting sources to zero and finding the resistance between the terminals. The short-circuit and open-circuit method seems a good choice.

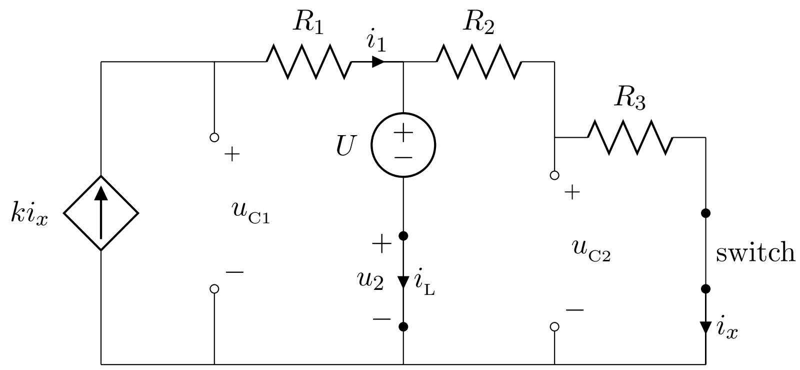
With a-b short-circuited, $u = 0$, and hence $i_x = 0$. KCL at the top node gives $i + ki = I$, hence $i_{\text{sc}} = \frac{I}{1+k} = I_{\text{N}}$.

With a-b open-circuited, $i = 0$, and so by KCL at the top node $I = i_x$ so $u_{\text{oc}} = IR_3$. The Norton resistance is then $\frac{u_{\text{oc}}}{i_{\text{sc}}} = R_3(1+k)$.

If the simplification of ignoring $R_1$, $R_2$ and $hi_x$ is not seen, some more work with equations will have been needed, e.g. defining a further potential below $R_2$, with a further KCL at that node. Instead of defining potentials (and a reference node) one can define one more voltage besides $u$, e.g. define the unknown voltage across the current-source $ki$, then eliminate this from the equations.

**4)**

**a)** Initial equilibrium: $t = 0^-$.
Draw the circuit in its state at time: switch closed.
Make all possible simplifications: replace $C$ and $L$ components, based on equilibrium with constant sources.

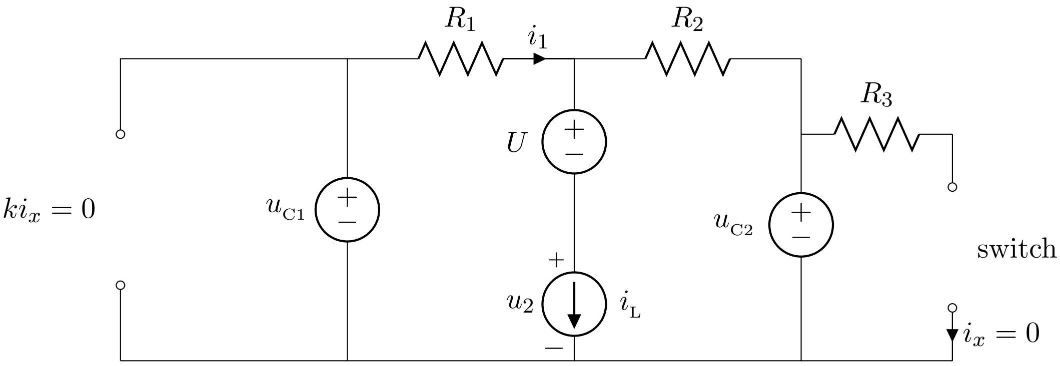



By normal dc analysis of this circuit,

$i_1(0^-) = \frac{kU}{R_2+R_3}$ from $i_1 = ki_x$ (KCL) and $i_x = \frac{U}{R_2+R_3}$ (KVL, right).

$u_2(0^-) = 0.$ as the inductor has zero voltage in equilibrium ($\frac{\mathrm{d}i}{\mathrm{d}t} = 0$).

**b)** Immediately after the change (switch opened): $t = 0^+$.
Again, draw the circuit in its current state: the switch is now open.
For $C$ and $L$ components, **continuity** provides the values of continuous variables, based on the equilibrium values at $t = 0^-$; these can be drawn as sources to ease our analysis.




The open switch ensures that $i_x = 0$, and hence the dependent current source $ki_x$ is fixed to zero, equivalent to an open circuit.

This remaining circuit has three branches between the top and bottom nodes: let's define the voltage between these two nodes as $u$. Writing KCL at the upper node (above source $U$),

$$\frac{u - u_{C1}}{R_1} + i_L + \frac{u - u_{C2}}{R_2} = 0,$$

from which

$$u = \frac{u_{C1}R_2 + u_{C2}R_1 - i_L R_1 R_2}{R_1 + R_2}.$$

The sought quantity $i_1$ can now be found as

$$i_1(0^+) = \frac{u_{C1} - u}{R_1} = \frac{u_{C1} - u_{C2} + i_L R_2}{R_1 + R_2}.$$

The other sought quantity is the voltage across the inductor. It is found from KVL, as

$$u_2(0^+) = u - U = \frac{u_{C1}R_2 + u_{C2}R_1 - i_L R_1 R_2}{R_1 + R_2} - U.$$

The above expressions are not yet in terms of the given (known) quantities, since they includes the capacitor and inductor states, such as $u_{C1}(0^+)$. These will have to be found, by assuming continuity from the equilibrium before the change.
At $t = 0^-$, looking back to the earlier circuit-diagram, we find that:

$$\begin{aligned} u_{C1}(0^-) &= U\left(1 + \tfrac{kR_1}{R_2+R_3}\right) &= u_{C1}(0^+) \\ u_{C2}(0^-) &= U\,\tfrac{R_3}{R_2+R_3} &= u_{C2}(0^+) \\ i_L(0^-) &= U\,\tfrac{k-1}{R_2+R_3} &= i_L(0^+) \end{aligned}$$

Substituting these, and simplifying: $i_1(0^+) = \frac{k}{R_2+R_3}U$, and $u_2(0^+) = 0$. Notice that these have turned out to be the same at $t =^+$ as at $t = 0^-$, although they are not continuous quantities. The reason is that the change (switch opening) and its immediate consequence (dependent-source becoming zero, as $i_x = 0$) are initially 'hidden' by the two capacitors. These capacitors are in parallel with the two outer branches of the circuit: their behaviour like voltage-sources at $t = 0^+$ means that the outer parts of the circuit are irrelevant to the central part where the marked $i_1$ and $u_2$ are. It's quite a lot of work to make the above algebraic steps, but on the other hand it's easy to fail to notice a more physically-based argument such as the above, that would have avoided the algebra.

**c)** Final equilibrium: $t \to \infty$.

The only difference here, compared to the initial equilibrium (part 'a'), is that the switch is open. This, however, makes the further simplification that $i_x = 0$ and therefore the dependent source is an open-circuit (as in part 'b').

$i_1(\infty) = 0$ KCL: the only paths for $i_1$ are zero current-source or a capacitor with $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$.

$u_2(\infty) = 0$ as the inductor has zero voltage in equilibrium ($\frac{\mathrm{d}i}{\mathrm{d}t} = 0$).

**5)**

The only change that happens in this circuit is the step-change in the voltage-source value: it changes from 0 to $U$ at time $t = 0$.

Let's try to find the continuous variable of the capacitor, as a time-function in the requested period of '$t > 0$'. We can call this variable (the capacitor voltage) $u(t)$, choosing the positive reference side at the top. Then the requested quantity $i(t)$ can be found when $u(t)$ is found.

The source-resistance (e.g. Thevenin resistance) seen by the capacitor is simply $R$. This can be seen by setting both sources to zero, removing the capacitor, and finding the resistance between the nodes where the capacitor was connected. Thus the time-constant for changes in this circuit is $CR$.

The final value $u(\infty)$ can be found by KVL. The equilibrium condition requires no current in the capacitor, so all of the current $I$ flows in the resistor $R$. By KVL in the left-hand loop, $u(\infty) = -U - IR$.

The initial value $u(0^+)$ can be found from continuity as being equal to the initial equilibrium value $u(0^-)$. The calculation is the same as for $u(\infty)$ except that we need 0 instead of $U$. Hence, $u(0^+) = u(0^-) = -IR$.

Putting these together, by the principle of $y(t) = y_{\text{final}} + (y_{\text{initial}} - y_{\text{final}})\, \mathrm{e}^{-t/\tau}$,

$$u(t) = -U - IR + (-IR + IR + U)\,\mathrm{e}^{-t/CR} = -IR - U\left(1 - \mathrm{e}^{-t/CR}\right).$$

The requested quantity was the capacitor's current, not its voltage. The defined directions of our chosen voltage and the marked current are such that

$$i(t) = -C\frac{\mathrm{d}u(t)}{\mathrm{d}t}.$$

Substituting the above solution for $u(t)$, the solution is $i(t) = \dfrac{U}{R}\,\mathrm{e}^{-t/CR}$.

Another way of solving this would be to use the initial and final values of the *current*, instead of working first with the capacitor's continuous variable of voltage. The danger in this is that the current can jump between different values at $t = 0^-$ and $t = 0^+$. One must be careful to use the current at $t = 0^+$ as the initial value of the variable. This current turns out to be $i(0^+) = U/R$ (in contrast to $i(0^-) = 0$); finding this requires use of the equilibrium voltage and continuity, so one does not avoid this step by working directly with the current. The final current is $i(\infty) = 0$. Putting these initial and final values together with the time-constant will also lead to the same solution as shown above.

As a further approach, the differential-equation method can be started from the relations

$$\begin{aligned} \frac{U + u(t)}{R} + I &= i(t) \quad \text{(KCL)} \\ -C\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= i(t) \quad \text{(capacitor).} \end{aligned}$$

Hence

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{1}{CR}u(t) = -\frac{1}{CR}\,(U + IR)\,,$$

for which the general solution can be found and the initial condition included.