KTH EI1102 / EI1100 Elkretsanalys Omtenta 2015-10-29 kl 14–19

Tentan har 6 tal i 2 delar: tre tal i del A (15p), tre i del B (15p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ...

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Godkänd tenta kräver minst 25% i del A, 25% i del B, och 50% i genomsnitt (båda delar). Betyget räknas från summan över båda delar, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

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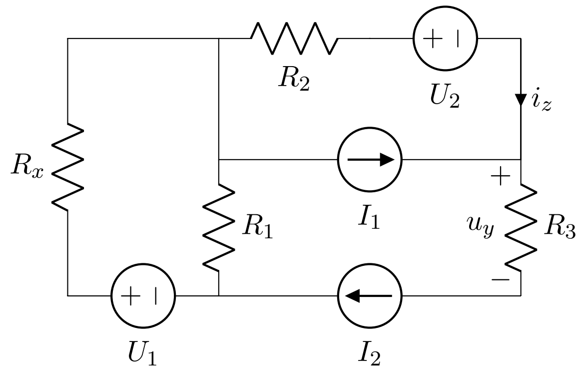
Del A. Likström och Transient

1) [5p]

a) [4p] Bestäm de följande:
De markerade u_y.
De markerade i_z.
Effekten som källan U_2 absorberar.
Effekten som motståndet R_1 absorberar.

b) [1p] Uttryck R_x som funktion av andra komponentvärden, sådant att den största möjliga effekten levereras till motståndet R_x från resten av kretsen.

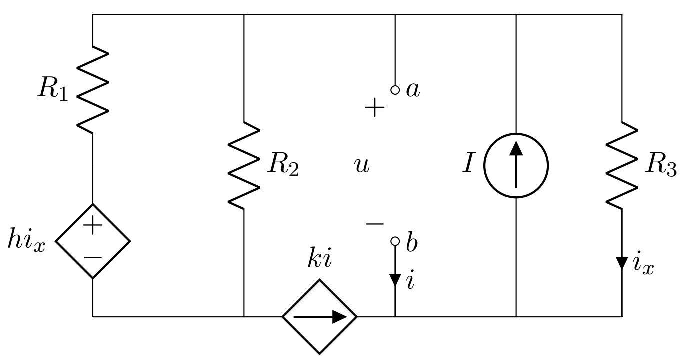


2) [5p]

Bestäm Nortonekvivalanten med avseende på polerna a och b.

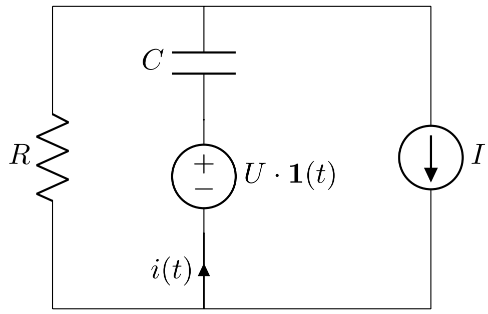
Kom ihåg konventionen att de markerade u, i och i_x är okända.

Ledning: Vilka komponenter är relevanta för lösningen?



3) [5p]

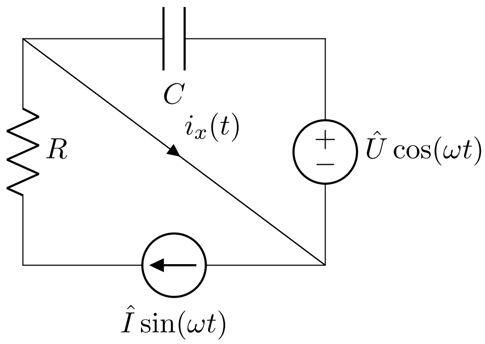
Bestäm $i(t)$ för tider $t > 0$.



Del B. Växelström

4) [5p]

Bestäm $i_x(t)$.

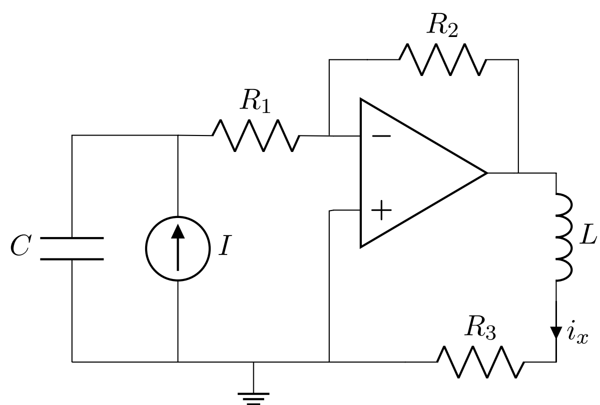


5) [5p]

a) [2,5p] Bestäm nätverksfunktionen $H(\omega) = \frac{i_x(\omega)}{I(\omega)}$, och visa att den kan skrivas som

$$H(\omega) = \frac{-k}{(1 + \mathrm{j}\omega/\omega_1)\,(1 + \mathrm{j}\omega/\omega_2)}.$$

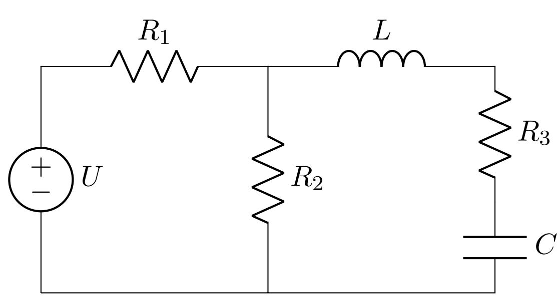
b) [2,5p] Skissa ett Bode amplituddiagram av $H(\omega)$, på antagandet att $\omega_1 \ll \omega_2$ och $k = 10$.
Markera viktiga punkter och lutningar.



6) [5p]

Källan ger en växelspänning med vinkelfrekvens ω, och effektivvärde U.

a) [3p] Vilket val av C gör att den största möjliga effekten levereras till motståndet R_3 från resten av kretsen? Uttryck C som funktion av andra komponentvärden.



b) [2p] När C är enligt svaret i deltal 'a', vad är effekten levererat till R_3?
(Om du inte har löst deltal 'a' kan du behandla i stället det allmänna fallet med kapacitans C.)

Solutions (EI1102/EI1100, HT15, 2015-10-29)

Q1 Note: this is the same as Q1 in the EI1110 TEN1 2015-10-29.

a)

$u_y = I_2 R_3$ series with current-source (i.e. KCL)

$i_z = I_2 - I_1$ KCL

$P_{\mathrm{U2}} = i_z U_2 = U_2(I_2 - I_1)$ note: this expression defines the power *into* source U_2. For some choices of component values the resulting number will be negative, indicating that in fact this source is supplying power to the rest of the circuit, and for other choices of component values the resulting number will be positive, indicating that the source is absorbing (receiving) power from the rest of the circuit. A source can absorb or produce power, whereas a resistor of positive value can only absorb it (the current is always going from a higher to lower potential, since this potential-change is what drives the current through the resistor).

$P_{\mathrm{R1}} = \left(\frac{U_1 - I_2 R_x}{R_x + R_1}\right)^2 R_1$ this can be found by nodal analysis on one node (i.e. KCL), which allows the voltage across, or current through, R_1 to be found. Looking at the node above (or below) R_1, there are three branches: U_1 and R_x in one branch, R_1 is another, and all the other components are equivalent to a source I_2. Hence, with u being the voltage across R_1, KCL gives $\frac{u-U_1}{R_x} + \frac{u}{R_1} + I_2 = 0$. We could solve for u, then find $P_{\mathrm{R1}} = u^2/R_1$. Or substitute $u = iR_1$ where i is the current in R_1, and thus solve KCL directly for the current, followed by $P_{\mathrm{R1}} = i^2 R_1$.

b) If R_x is to be adjusted to receive the highest possible power from the rest of the circuit, it should have the same value as the Thevenin resistance of the rest of the circuit to which it's connected.

As there are no dependent sources, our easiest way to find the Thevenin resistance (between the nodes where R_x connects, without R_x present) is to set the sources to zero. The part to the right then disappears, due to the series current-sources (open-circuits). The Thevenin resistance is just R_1.

$R_{x(\mathrm{maxP})} = R_1$.

Q2 Note: this is the same as Q3 in the EI1110 TEN1 2015-10-29.

$I_{\mathrm{N}} = \frac{I}{1+k}$ and $R_{\mathrm{N}} = (1+k)R_3$.

The Norton source should be drawn, showing the current-source pointing towards the 'a' terminal (or else I_{N} should be a negative expression) and showing the resistance in parallel with the source.

One method. From the terminals a-b, everything to the left is equivalent to the dependent current-source ki (it's a branch including a series current-source). So KCL at the top node gives $ki + i - I + \frac{u}{R_3} = 0$.

At its terminals a Norton source has the u-i relation $i = I_{\mathrm{N}} - \frac{u}{R_{\mathrm{N}}}$. By rearranging the KCL to group the terms in a similar way, we can compare the i-u relation of the circuit to the i-u relation of a generic Norton source,

$$i = \frac{I}{1+k} - \frac{1}{(1+k)R_3}\,u = I_{\mathrm{N}} - \frac{u}{R_{\mathrm{N}}}.$$

from which the Norton parameters are identified.

Another method. There are dependent sources in the circuit, so we *cannot* rely on the method of finding the circuit's equivalent resistance by setting sources to zero and finding the resistance between the terminals. The short-circuit and open-circuit method seems a good choice.

With a-b short-circuited, $u = 0$, and hence $i_x = 0$. KCL at the top node gives $i + ki = I$, hence $i_{\mathrm{sc}} = \frac{I}{1+k} = I_{\mathrm{N}}$.

With a-b open-circuited, $i = 0$, and so by KCL at the top node $I = i_x$ so $u_{\text{oc}} = IR_3$. The Norton resistance is then $\frac{u_{\text{oc}}}{i_{\text{sc}}} = R_3(1+k)$.

If the simplification of ignoring R_1, R_2 and hi_x is not seen, some more work with equations will have been needed, e.g. defining a further potential below R_2, with a further KCL at that node. Instead of defining potentials (and a reference node) one can define one more voltage besides u, e.g. define the unknown voltage across the current-source ki, then eliminate this from the equations.

Q3 Note: this is the same as Q5 in the EI1110 TEN1 2015-10-29.

The only change that happens in this circuit is the step-change in the voltage-source value: it changes from 0 to U at time $t = 0$.

Let's try to find the continuous variable of the capacitor, as a time-function in the requested period of '$t > 0$'. We can call this variable (the capacitor voltage) $u(t)$, choosing the positive reference side at the top. Then the requested quantity $i(t)$ can be found when $u(t)$ is found.

The source-resistance (e.g. Thevenin resistance) seen by the capacitor is simply R. This can be seen by setting both sources to zero, removing the capacitor, and finding the resistance between the nodes where the capacitor was connected. Thus the time-constant for changes in this circuit is CR.

The final value $u(\infty)$ can be found by KVL. The equilibrium condition requires no current in the capacitor, so all of the current I flows in the resistor R. By KVL in the left-hand loop, $u(\infty) = -U - IR$.

The initial value $u(0^+)$ can be found from continuity as being equal to the initial equilibrium value $u(0^-)$. The calculation is the same as for $u(\infty)$ except that we need 0 instead of U. Hence, $u(0^+) = u(0^-) = -IR$.

Putting these together, by the principle of $y(t) = y_{\text{final}} + (y_{\text{initial}} - y_{\text{final}})\,\mathrm{e}^{-t/\tau}$,

$$u(t) = -U - IR + (-IR + IR + U)\,\mathrm{e}^{-t/CR} = -IR - U\left(1 - \mathrm{e}^{-t/CR}\right).$$

The requested quantity was the capacitor's current, not its voltage. The defined directions of our chosen voltage and the marked current are such that

$$i(t) = -C\frac{\mathrm{d}u(t)}{\mathrm{d}t}.$$

Substituting the above solution for $u(t)$, the solution is $i(t) = \dfrac{U}{R}\,\mathrm{e}^{-t/CR}$.

Another way of solving this would be to use the initial and final values of the *current*, instead of working first with the capacitor's continuous variable of voltage. The danger in this is that the current can jump between different values at $t = 0^-$ and $t = 0^+$. One must be careful to use the current at $t = 0^+$ as the initial value of the variable. This current turns out to be $i(0^+) = U/R$ (in contrast to $i(0^-) = 0$); finding this requires use of the equilibrium voltage and continuity, so one does not avoid this step by working directly with the current. The final current is $i(\infty) = 0$. Putting these initial and final values together with the time-constant will also lead to the same solution as shown above.

As a further approach, the differential-equation method can be started from the relations

$$\begin{aligned} \frac{U + u(t)}{R} + I &= i(t) \quad \text{(KCL)} \\ -C\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= i(t) \quad \text{(capacitor)}. \end{aligned}$$

Hence

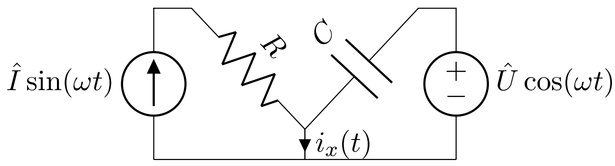
$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{1}{CR}u(t) = -\frac{1}{CR}(U + IR),$$

for which the general solution can be found and the initial condition included.

Q4

$i_x(t) = \left(\hat{I} - \omega C\hat{U}\right)\sin(\omega t)$.

It might help to re-draw this, to make clear that the two sides are independent of each other: they are two loops, with the bottom node as a common node, and the marked $i_x(t)$ is the sum of the currents in these two loops.



The resistor is in series with the current source, so its current is already determined (the resistance R is not relevant). The capacitor has a voltage $\hat{U}\cos(\omega t)$ across it, which determines its current.

For more complicated circuits with ac sources, it would be very helpful to use ac analysis ('jω-metoden'). We could choose e.g. a cosine reference, and describe the sources as $-\mathrm{j}\hat{I}$ and $\hat{U}$. Then the current in the capacitor (from the source towards the point marked i_x) is $\mathrm{j}\omega C\hat{U}$. The current in the resistor, towards this same point, is $-\mathrm{j}\hat{I}$. The total marked current is thus $i_x(\omega) = \mathrm{j}\left(\omega C\hat{U} - \hat{I}\right)$. Converting back to a time-function, notice that a positive imaginary value (j) of the phasor corresponds to $-\sin(\omega t)$. Hence,

$$i_x(t) = \left(\omega C\hat{U} - \hat{I}\right)(-\sin(\omega t)) = \left(\hat{I} - \omega C\hat{U}\right)\sin(\omega t).$$

In our specific case, each part of the analysis is independent and relatively simple. The current in the resistor is $\hat{I}\sin(\omega t)$. The capacitor and voltage source are directly connected, so they have the same voltage, $\hat{U}\cos(\omega t)$. For a capacitor, $i = C\frac{\mathrm{d}u}{\mathrm{d}t}$, from which our capacitor carries a current $\omega CU(-\sin(\omega t))$. Since these two currents both are a constant times a $\sin(\omega t)$ function, their sum is easily written. This sum is the solution, as both currents return via the marked point of i_x.

Q5

a)

The opamp's inverting input is a "virtual earth", i.e. it has zero potential. That is because the feedback holds it to the same potential as the non-inverting input, which is directly earthed.

In the node above the capacitor, we can define potential v, and can use nodal analysis to find this from KCL:

$$\mathrm{j}\omega Cv - I + \frac{v}{R_1} = 0 \qquad \Longrightarrow \qquad v = \frac{I}{\frac{1}{R_1} + \mathrm{j}\omega C}.$$

The combination of the opamp, R_1, and R_2, can be seen as a classic inverting amplifier, where $\frac{v_{\text{out}}}{v_{\text{in}}} = -R_2/R_1$. Now that we know v_{in}, the opamp's output potential can be found,

$$v_{\text{out}} = -\frac{R_2}{R_1}\frac{I}{\frac{1}{R_1} + \mathrm{j}\omega C}.$$

The sought current, i_x, is then

$$i_x = \frac{v_{\text{out}}}{R_3 + \mathrm{j}\omega L} = -\frac{R_2}{R_1}\frac{I}{\frac{1}{R_1} + \mathrm{j}\omega C}\frac{1}{R_3 + \mathrm{j}\omega L}.$$

Then the network function is

$$H(\omega) = \frac{i_x(\omega)}{I(\omega)} = \frac{-R_2/R_1}{\left(\frac{1}{R_1} + \mathrm{j}\omega C\right)(R_3 + \mathrm{j}\omega L)} = \frac{-R_2/R_3}{(1 + \mathrm{j}\omega CR_1)(1 + \mathrm{j}\omega L/R_3)}.$$

This fits the given form of equation if $k = R_2/R_3$, $\omega_1 = \frac{1}{CR_1}$, and $\omega_2 = R_3/L$; alternatively, the definitions of ω_1 and ω_2 could be swapped, as this was not defined.

Q6

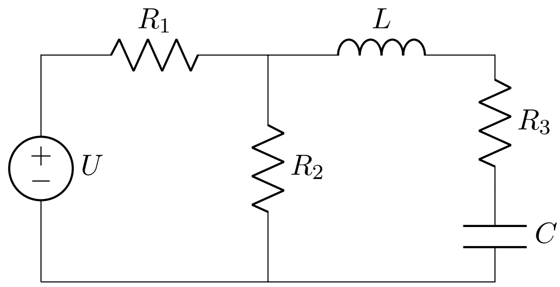
a)

This is *not* quite the classic maximum power question.

The classic is that there is a source with fixed parameters, and a load that can be varied. The source parameters are its Thevenin or Norton equivalent values: a source impedance, and an open-circuit voltage or short-circuit current. These parameters determine the maximum power that the source can supply. The load parameters are two values of magnitude and angle (or rectangular form) of an impedance or of a voltage or current source. These determine whether the load is giving the right conditions to extract maximum power from the source. As we should remember, the condition for maximum power into a load impedance is that this should be the complex conjugate of the source impedance. For any other type of load that's not an impedance, for example a current-source, it should result in the same voltage and current as this choice of impedance would.

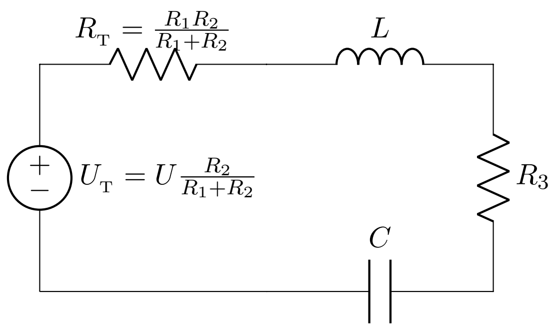
As a comment for students in VT19, who asked for further explanation of this solution: I think I'm now too cowardly to put this question in an exam, as it could seem a bit of a trick when you've only really seen classic types of maximum power question in our examples. It might have been expected that students taking this exam would not bother thinking much about whether the maximum power principle they'd learned was strictly applicable to this situation.
It should be possible to work it out from your existing knowledge, such as from the brief way in which we argued our way to the ac maximum power crierion. But it could be confusing to some as we haven't really thought of other types of case like when one can vary only one parameter.

In this question we're trying to get the maximum possible power into the resistor R_3, but we are only permitted to adjust the capacitor. Changing C will change the power to R_3, but we would have to be able to change R_3 as well if wanting the power to R_3 to be as high as the maximum possible power that the rest of the circuit could supply. This question is only about the maximum possible for a fixed R_3.



Let's start with one method; then we can continue to consider alternative methods. We can replace the voltage-source and two adjacent resistors with their Thevenin equivalent, as shown below.

What determines the power into R_3? As we have a single loop, and therefore a single current, it is convenient to express the power in terms of current. This is $|i|^2 R_3$, where i is the current round the loop. Since in this problem R_3 is fixed, anything we can change that increases the current magnitude will increase the power to R_3.



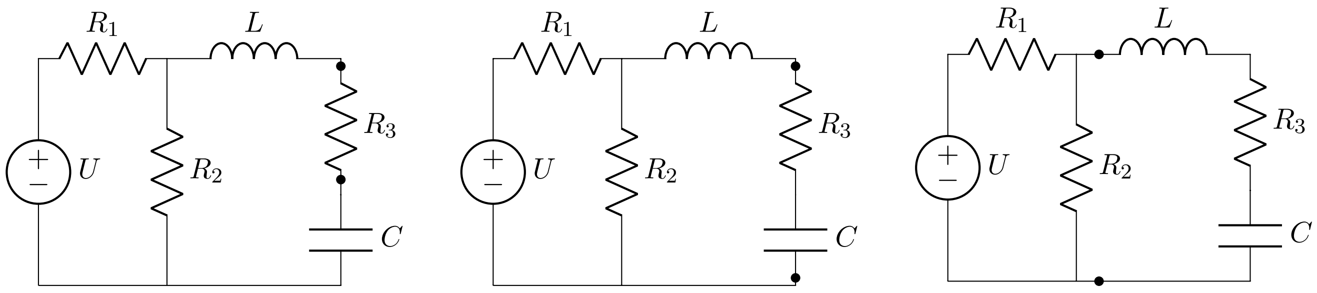
The only thing we are permitted to change is C. The total current is

$$i = \frac{U_\mathrm{T}}{Z_\mathrm{total}} = \frac{U\frac{R_2}{R_1+R_2}}{\frac{R_1R_2}{R_1+R_2} + R_3 + \mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C}}.$$

Anything that can make the magnitude of the denominator (nämnare) smaller is helping to make the current magnitude larger, and thus the power to R_3 greater. We are only allowed to change C. Clearly, this allows us to make the imaginary part become zero, by setting C such that $\mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C} = 0$. That is the best we can do; we can't adjust the real part, and any positive or negative imaginary part will increase the magnitude of the total impedance.

As another type of solution, we could make simplistic use of the principle that maximum power is when $Z_{\text{load}} = Z^*_{\text{source}}$. By 'simplistic' it is meant that one interprets the principle to mean that the imaginary parts of the source and load impedance should be made to be opposite, even if the real parts can't be adjusted. (Although we've shown above that this is true, it's not safe to assume that a theorem can be separately to one parameter even when another parameter is fixed and not at its optimal value.)

We can choose the 'load' to be just R_3, or this together with the capacitor and/or the inductor. Some of these options are shown by the positions of the terminals in the circuits below.



The load must *include* R_3, as this is what the power will be delivered to. The maximum power theorem is about maximising power out of the source, to the load. It cannot include R_1 or R_2, as these are what are consuming active power within the 'source' and thus limiting what can go to the load. The reactive components, L and C, can be in either the load or the source, since they cannot consume any active power of their own, and therefore their presence in the load does not change that the 'load' active power equals the active power into R_3, which we have to maximise.

In any of these cases, attempting to make the imaginary parts of the source and load be 'equal and opposite' results in choosing $\mathrm{j}\omega L - \mathrm{j}\frac{1}{\omega C} = 0$ and thus $C = \frac{1}{\omega^2 L}$. That is the same result as we got earlier. It will also turn out

b)

The first solution method used for part 'a' showed that the maximum power condition is when

$$\mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C} = 0$$

and derived the current i around the loop with U, R_1 and R_2 replace by their Thevenin equivalent.

The power into R_3 is then $|i|^2 R_3$, which is

$$P_{\text{R3}} = \left(\frac{\frac{U R_2}{R_1 + R_2}}{\frac{R_1 R_2}{R_1 + R_2} + R_3 + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C}} \right)^2 R_3 = \frac{U^2 R_2^2 R_3}{(R_1 R_2 + R_1 R_3 + R_2 R_3)^2}$$