EI1110 Elkretsanalys (Elektro) Omtenta TEN1, 2016-01-07 kl 08–13

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt.

Tentan har 5 tal i två sektioner: 3 i sektion A (12p), och 2 i sektion B (10p).

Godkänd tentamen TEN1 kräver:

$$\frac{\max(a, a_\mathrm{k})}{A} \geq 40\% \qquad \& \qquad \frac{b}{B} \geq 40\% \qquad \& \qquad \frac{\max(a, a_\mathrm{k}) + b}{A + B} \geq 50\%$$

där A=12 och B=10 är de maximala möjliga poängen från sektionerna A och B, a och b är poängen man fick i dessa respektive sektioner i tentan, och a_k är poängen man fick från kontrollskrivning KS1 vilken motsvarar tentans sektion A; funktionen max() tar den högre av sina argument.

Betyget räknas från summan över båda sektioner, igen med bästa av sektion A och KS1, $\frac{\max(a,a_\mathrm{k})+b}{A+B}$. Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

I vissa gränsfall där betyget är lite under 50%, eller bara en av sektionerna är underkänd trots 50% eller bättre totalt, kommer betyget 'Fx' registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara **kända** storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller en spänningskälla) antas vara **okända** storheter.

Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner.

Dela tiden mellan talen — senare deltal brukar vara svårare att tjäna poäng på ... fastna inte!

Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Använd kvarstående tid för att kontrollera svaren. Lycka till!

Nathaniel Taylor (073 919 5883)

Sektion A. Likström

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1) [4p]

Bestäm effekten absorberad av följande komponenten:

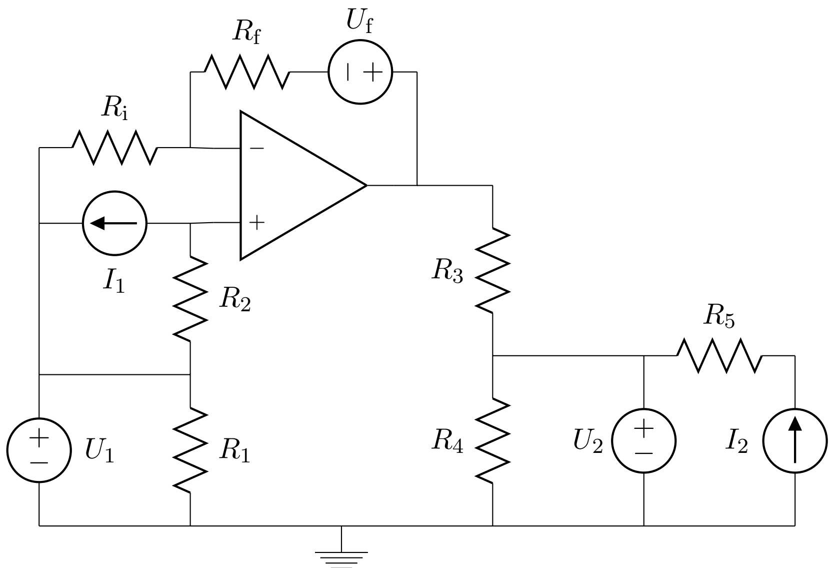
a) [1p] Motstånden R_1 och R_2.

b) [1p] Motstånden R_4 och R_5.

c) [1p] Källan I_1.

d) [1p] Motståndet R_3.

Ledning: I vissa fall ovan är lösningen beroende bara på ett fåtal närliggande komponenter.

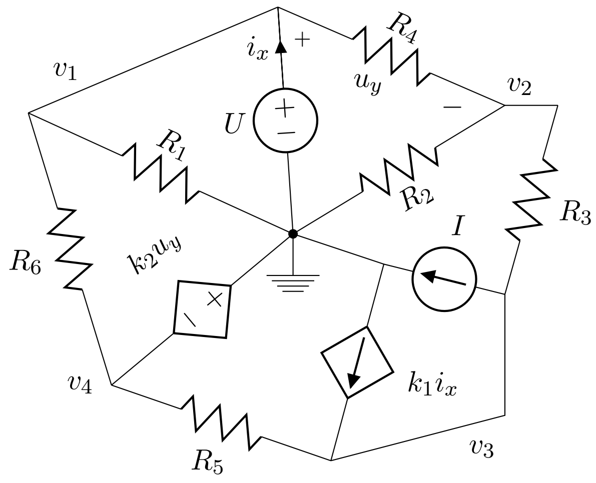


2) [4p]

Skriv ekvationer som skulle kunna lösas för att finna de markerade potentialerna v_1, v_2, v_3, v_4 som funktioner av de givna komponentvärderna.

Du **måste inte** lösa eller förenkla dina ekvationer.

Använd helst en systematisk metod, för att försäkra tillräckliga ekvationer utan onödigt arbete.



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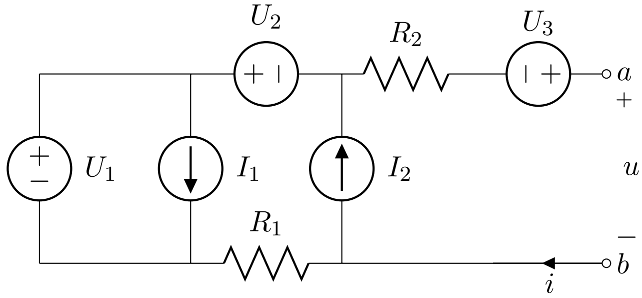
3) [4p] Kända storheterna: U och R, där

$$U_1 = U_2 = U_3 = U$$
$$R_1 = R_2 = R$$
$$I_1 = I_2 = U/R$$

Bestäm en krets med bara två komponenter, som kan ersätta alla komponenterna här och ge identiskt beteende av u och i med avseende på polerna a-b.



Skriv svaret med diagram, där komponenterna definieras **bara med** storheterna U och R.

Ledning: Förenkla. Observera att det inte finns några beroende källor här.

Sektion B. Transient

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4) [5p]

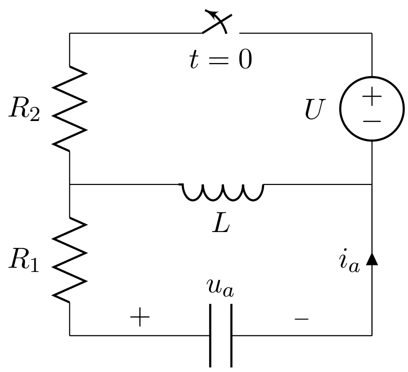
Bestäm följande värden:

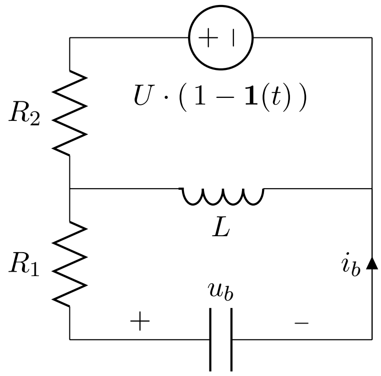
a) [2p] $u_a(0^-)$, $i_b(0^-)$.

b) [2p] $i_a(0^+)$, $i_b(0^+)$.

c) [1p] $i_a(\infty)$, $u_b(\infty)$.

Obs: '$\mathbf{1}(t)$' är enhetsstegfunktionen.





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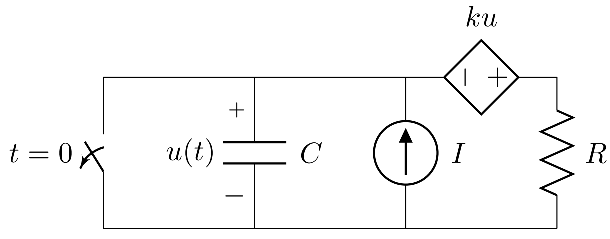
5) [5p]

Bestäm $u(t)$ för tider $t > 0$.



Solutions, EI1110 TEN1 2016-01-07

1)

a)

$P_{\mathrm{R1}} = \frac{U_1^2}{R_1}$ parallel with voltage-source (rest of circuit is irrelevant).

$P_{\mathrm{R2}} = I_1^2 R_2$ series with current-source (as opamp input has no current).

b)

$P_{\mathrm{R4}} = \frac{U_2^2}{R_4}$ parallel with voltage-source.

$P_{\mathrm{R5}} = I_2^2 R_5$ series with current-source.

c)

$P_{\mathrm{I1}} = -I_1^2 R_2$ KVL around I_1 & R_2 (knowing that the current in R_2 is the current I_1).
Note the negative sign: the question was about the power absorbed by the component; when taking account of the direction of current and voltage we see the current source is supplying $I_1^2 R_2$.
Another approach is to re-draw: after ignoring the opamp input (which has no current), you will see that I_1 and R_2 are directly connected to each other and have only one node connecting to the rest of the circuit, which therefore becomes irrelevant.

d)

This is more difficult. The resistor R_3 is connected between two potentials that do not depend on the value R_3 ... this would be an easy solution if the potentials were known, but one of them is not yet known. That unknown is the output voltage of the opamp, which we only can find after solving the circuit of the inputs and the feedback.

At the non-inverting input, the potential can be found by adding voltages along a path from the earth-node: $v_+ = U_1 - I_1 R_2$ (you can also see this as a form of KVL, where the final voltage in the KVL loop is the unmarked one from the +-input back to earth).

Let's call the potential of the opamp's output v. The inverting input is assumed to have the same potential as the non-inverting input: $v_- = v_+$. KCL at the inverting input then gives

$$\frac{U_1 - v_-}{R_\mathrm{i}} + \frac{(v - U_\mathrm{f}) - v_-}{R_\mathrm{f}} = 0 \quad \Longrightarrow \quad \frac{U_1 - (U_1 - I_1 R_2)}{R_\mathrm{i}} + \frac{(v - U_\mathrm{f}) - (U_1 - I_1 R_2)}{R_\mathrm{f}} = 0$$

from which

$$v = U_1 + U_\mathrm{f} - I_1 R_2 \left(1 + \frac{R_\mathrm{f}}{R_\mathrm{i}}\right)$$

At the bottom of R_3 the potential is U_2, due to the direct connection to that voltage source, which is connected to earth on the other side. The voltage across R_3 is therefore $v - U_2$.

$$P_{\mathrm{R3}} = \frac{(v-U_2)^2}{R_3} = \left(U_1 + U_\mathrm{f} - I_1 R_2 \left(1 + \frac{R_\mathrm{f}}{R_\mathrm{i}}\right) - U_2\right)^2 \Big/ R_3.$$

2) Two solution methods are shown, and a numerical check is made.

Extended nodal analysis ("the simple way")

Define the unknown current in voltage source U: we'll call it i_α, going into the source's + terminal.[1] Likewise, define i_β into the + terminal of the dependent voltage source $k_2 u_y$.

KCL (outgoing currents) at all nodes except ground:

$$\text{KCL}(1): \quad 0 = i_\alpha + \frac{v_1}{R_1} + \frac{v_1 - v_4}{R_6} + \frac{v_1 - v_2}{R_4} \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + \frac{v_2 - v_1}{R_4} + \frac{v_2 - v_3}{R_3} \tag{2}$$

$$\text{KCL}(3): \quad 0 = I - k_1 i_x + \frac{v_3 - v_2}{R_3} + \frac{v_3 - v_4}{R_5} \tag{3}$$

$$\text{KCL}(4): \quad 0 = -i_\beta + \frac{v_4 - v_1}{R_6} + \frac{v_4 - v_3}{R_5} \tag{4}$$

Next, include the further information given by the voltage sources,

$$v_1 = U \tag{5}$$

$$v_4 = -k_2 u_y \tag{6}$$

Define the marked quantities that are the controlling variables of the dependent sources. These should be defined in terms of variables (known or unknown) that we already have in the equation system.

$$i_x = -i_\alpha \qquad \text{or} \quad i_x = \frac{v_1}{R_1} + \frac{v_1 - v_4}{R_6} + \frac{v_1 - v_2}{R_4} \tag{7}$$

$$u_y = v_1 - v_2 \tag{8}$$

Now there are 8 equations, in the 8 unknowns: there are 4 unknown node-potentials (and 4 KCL equations); then 2 unknown currents in voltage-sources (and 2 equations relating node-potentials in terms of source-voltages); and 2 unknown marked controlling variables (with 2 equations expressing them in terms of other quantities).

So that's it. The above equations are a sufficient answer.

In this special case where the current in the voltage-source U has been described both as i_x and as i_α (but with opposite reference directions) the relation in (7) can be simply written $i_x = -i_\alpha$, or written using KCL at node 1. From equation (1), these two forms can be seen to be equivalent.

Alternative: Supernode method

KCL is done at each node (or supernode group) apart from the ground node.

There are two voltage-sources, U and $k_2 u_y$. One terminal of each is connected to the earth node, and the other terminals are connected to nodes 1 and 4 respectively. We can say that these two nodes and the earth node form a supernode. One part of the supernode (the earth node) has a known potential (zero). Thus, the marked potentials v_1 and v_4 can be written as U and $k_2 u_y$. KCL is only needed at nodes 2 and 3, since it is not used at the earth node or at any part of a supernode that contains the earth node. (This reduction in the number of KCLs is obtained by not caring about solving for currents in the voltage sources.)

If we want to write the equations purely in terms of the variables v_2 and v_3, we must eliminate the marked controlling variables i_x and u_y by defining them in terms of known quantities or v_2 or v_3, then

[1] In this particular circuit there already is a marked current i_x in the voltage source, which we could have used instead of defining a new unknown of i_α. But we'll try doing this according to the general set of rules we've learned, in which we introduce a new unknown for the current. It will be clear from the equations that we could just eliminate one of these currents by the substitution $i_x = -i_\alpha$.

substituting these definitions instead of i_x or u_y. It's easiest to handle u_y: it is just $U - v_2$, both of which are already defined (U is known, v_2 unknown). There is some more work in i_x: we can only find the current in a voltage source by looking at the rest of the circuit: in this case the current is the sum for the three resistors connecting to source U, and R_6 connects to node 4 whose potential now has to be written $k_2(U - v_2)$ if we want to have only v_2 and v_3 (not u_y) as unknowns in our supernode equations. Hence, $i_x = \frac{U}{R_1} + \frac{U-v_2}{R_4} + \frac{U-k_2(U-v_2)}{R_6}$.

The KCL equations at the supernode and other nodes (other than earth) are

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + \frac{v_2 - U}{R_4} + \frac{v_2 - v_3}{R_3} \tag{1}$$

$$\text{KCL}(3): \quad 0 = \frac{v_3 - k_2(U - v_2)}{R_5} + \frac{v_3 - v_2}{R_3} + I - k_1\left(\frac{U}{R_1} + \frac{U - v_2}{R_4} + \frac{U - k_2(U - v_2)}{R_6}\right) \tag{2}$$

It is not sufficient to answer with just the above equations, without also saying how to find the remaining two potentials:

$$v_1 = U \tag{3}$$

$$v_4 = k_2(U - v_2) \tag{4}$$

The above equations are a sufficient answer.

As usual, we can point out that the above 4 equations should have been able to be obtained by subsitution in the 8 equations.

3)

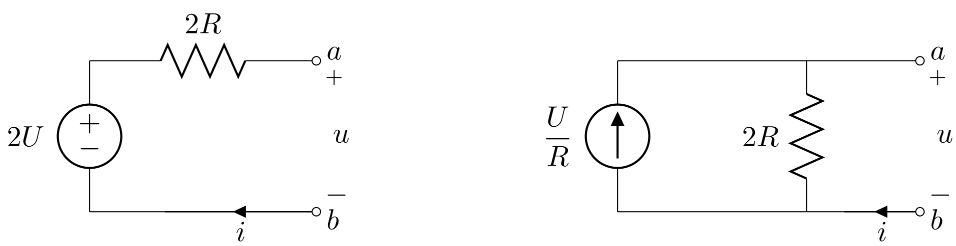
The question requests an equivalent with two components to be found: i.e. a Thevenin or Norton equivalent. The components are marked with unique symbols, but we are told that these can all be expressed in terms of U and R. In the following calculation we'll use the unique symbols for clarity, then simplify at the end.

The open-circuit voltage is $u_{i=0} = U_1 + U_3 - U_2 + I_2 R_1$. This can be found from KVL around the outer loop. The condition $i = 0$ implies zero voltage across R_2, and that all of I_2 must pass through R_1, from which the voltages across the resistors can be found.

As there are no dependent sources, the resistance can be found directly by zeroing the sources and finding the resistance between the terminals. In this case the circuit reduces to a series loop with just R_1 and R_2. The equivalent circuit's resistance is thus $R = R_1 + R_2$.

We can therefore make a Thevenin equivalent with $U_{\text{T}} = U_1 + U_3 - U_2 + I_2 R_1$ and $R_{\text{T}} = R_1 + R_2$.

Alternatively, a Norton equivalent can easily be found by source-transformation, where $I_{\text{N}} = \frac{U_{\text{T}}}{R_{\text{T}}} = \frac{U_1+U_3-U_2+I_2R_1}{R_1+R_2} = \frac{U}{R}$ and $R_{\text{N}} = R_1 + R_2 = 2R$.



Either one of the above circuits is a sufficient answer.

Alternative

The resistance could be found in a different way, by finding the open-circuit voltage $u_{i=0}$ as above, and the short-circuit current $i_{u=0}$: their ratio is the resistance, $R_{\text{N}} = R_{\text{T}} = \frac{U_{\text{T}}}{I_{\text{N}}} = \frac{u_{i=0}}{i_{u=0}}$. This would be well suited to a circuit containing dependent sources, where the method used above (setting sources to zero) is not sufficient.

To find the short-circuit current directly, we could remove the irrelevant I_1 (parallel with voltage-source U_1), then short-circuit the terminals a and b, and solve for i.

This can be done by writing KCL for the node at the top (or the bottom) of source I_2. In order to write the currents in the resistor branches we could define a voltage across source I_2 (or define the bottom as earth and the top as a potential); then i can be calculated after finding this voltage or potential. Alternatively, we can use the current i directly as the unknown, as the voltage across I_2 can be expressed as $iR_2 - U_3$:

$$0 = i - I_2 + \frac{(iR_2 - U_3) - U_1 + U_2}{R_1}$$

whence

$$i = I_{\text{N}} = \frac{U_1 + U_3 - U_2 + I_2 R_1}{R_1 + R_2}.$$

Alternative

Another possibility is to find an equation relating u and i, then to identify the source and resistance values of a Thevenin or Norton equivalent.

A very similar KCL approach to the above could be used, but with the unknown voltage u in the right-hand branch of the circuit,

$$i = \frac{U_1 + U_3 - U_2 + I_2 R_1 - u}{R_1 + R_2} = \frac{2U - u}{2R}.$$

For comparison, the equation for a Thevenin source (with the same directions of u and i as in our circuit) is $u = U_{\text{T}} - iR_{\text{T}}$, and for a Norton source $i = I_{\text{N}} - \frac{u}{R_{\text{N}}}$. The parameters U_{T}, I_{N} and $R_{(\text{N}|\text{T})}$ can be found by comparing these expressions with the circuit's expression for i and u.

4)

Notice the distinction between the two circuits: a voltage source becoming zero is not the same as a voltage source being disconnected. At times $t < 0$ both are equivalent. At times $t > 0$ the second circuit (right) has three parallel branches, but the first circuit (left) has only two, since the top branch is open.

a) $u_a(0^-) = 0$ and $i_b(0^-) = 0$.
The inductor L is in parallel with the branch containing the capacitor C, where u_a and i_b are marked. In equilibrium, with constant source values, we assume the inductor will have zero voltage, as we expect $\frac{\mathrm{d}i}{\mathrm{d}t} = 0$. This means that the whole branch of C and R_1 is connected to a short circuit: hence $u_a(0^-) = 0$. In equilibrium we also assume no *current* to flow in the capacitor, hence $i_b(0^-) = 0$.

b) $i_a(0^+) = \frac{-U}{R_2}$, and $i_b(0^+) = \frac{-U}{R_1+R_2}$.
This is the hardest part. At $t = 0^-$ the inductor carries a current U/R_2 from left to right, and the capacitor has zero voltage. By continuity, these states are still true at $t = 0^+$. In the first circuit the switch is open so the inductor's current can only flow through the lower branch, where i_a is marked. In the second circuit, the inductor's current can divide between the top and bottom branches. Remember: current division is *only valid for parallel resistors*, not for cases where there's something else like a voltage source in series with one or both resistors ... but in this case the voltage source has zero value, and the capacitor is known to be behaving like a zero voltage source at this time $t = 0^+$; thus, the source and capacitor can both be represented as short-circuits, making the resistors be in parallel. If you didn't feel happy to apply current division, you could have used KVL and KCL directly – that's how we derived current division, after all! If you used current division and didn't even think whether it was valid, then be *careful* in the future, as it's a common mistake to apply current division in cases where it's not valid!

c) $i_a(\infty) = 0$, and $u_b(\infty) = 0$.
Again (as in part '4a'), this is an equilibrium with constant sources, so the inductor has zero voltage. In this case, we also know that there *are* no sources (except a zeroed source in the second circuit): as there are resistors that will remove any stored energy in the capacitor or inductor we can also be confident from energy considerations that currents and voltages in the circuit will decay to zero.

5)

Until $t = 0$ the switch is short-circuiting the capacitor and current-source: therefore $u(0^-) = 0$, and by continuity of the capacitor's voltage, $u(0^+) = 0$.

After the switch opens, KCL in the node above the capacitor gives

$$0 = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} - I + \frac{u + ku}{R} \qquad \Longrightarrow \qquad \frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{1+k}{RC}u = \frac{I}{C}$$

This has a general solution of

$$u(t) \;=\; \frac{IR}{1+k} + U_\Delta \mathrm{e}^{-\frac{1+k}{RC}t}$$

where U_Δ needs to be found to fit the initial conditions. Recall that we know the state at $t = 0$, i.e. $u(0^+) = 0$; at this time the expoential term is $\mathrm{e}^0 = 1$. Thus,

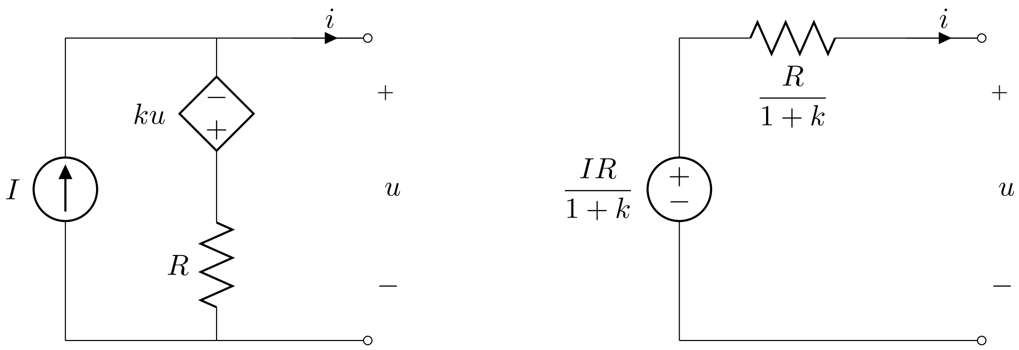
$$u(0) \;=\; 0 = \frac{IR}{1+k} + U_\Delta \qquad \Longrightarrow \qquad U_\Delta = -\frac{IR}{1+k},$$

from which the specific equation for $u(t)$ is

$$u(t) \;=\; \frac{IR}{1+k}\left(1 - \mathrm{e}^{-\frac{1+k}{RC}t}\right) \qquad (t > 0).$$

Alternative

The other way to do it is to find a two-terminal equivalent, at the capacitor terminals, of everything except the capacitor, for times $t > 0$.



If the equivalent (above right) has a stored charge of zero and is connected to a capacitor C at time zero, the time-function $u(t)$ will be the same as in our original question.

The final value is seen to be $u(\infty) = I\,\frac{R}{1+k}$.
The time-constant is $\frac{R}{1+k}C$.
The initial value is determined by the capacitor's initial charge: $u(0^+) = 0$.

From these the same time-function of $u(t)$ can be found as from the direct differential-equation method.

Notice that, if the dependent source had been the other way up, or if k were negative, it would be possible for the circuit to exhibit a negative equivalent resistance, leading to an unstable voltage u (never reaching steady state).