EI1120 Elkretsanalys, Kontrollskrivning KS2, 2016-02-19 kl 08–10

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt.

Kontrollskrivningen har 2 tal, med totalt 10 poäng. Den omfattar ämnet 'Transienter' och motsvarar sektion B i tentan. I tentan är kravet för godkänd minst 40% för sektion B, samt minst 50% över hela tentan.

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter.
Lösningar ska uttryckas i kända storheter, och förenklas.
Var tydlig med diagram och definitioner av variabler.
Dela tiden mellan talen — senare deltal brukar vara svårare att tjäna poäng på . . . fastna inte på dessa.
Kontrollera svarens rimlighet genom t.ex. dimensionskontroll eller alternativ lösningsmetod.

Använd återstående tid för att kontrollera svaren! Nathaniel Taylor (073 919 5883)

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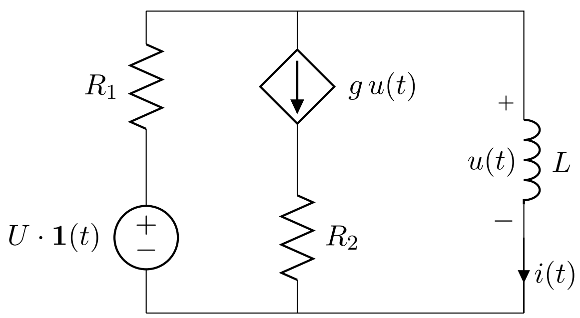
1) [5p]

Bestäm $i(t)$, för $t > 0$.



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2) [5p]

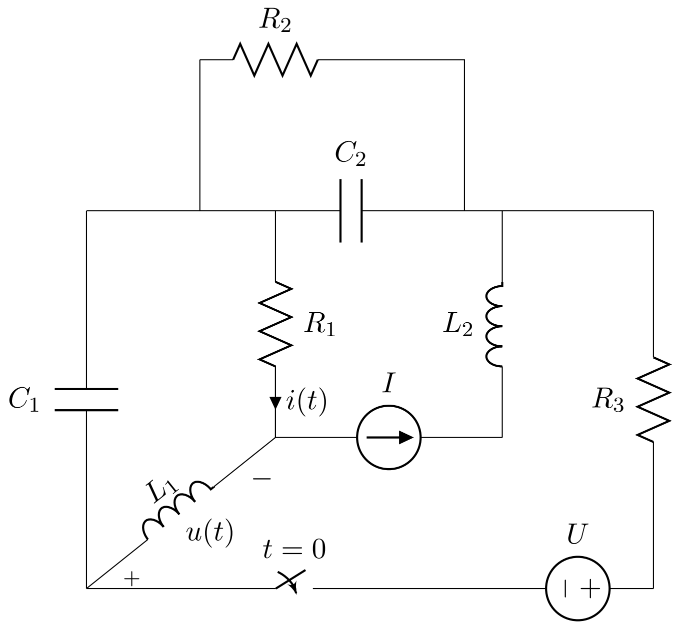
Storheterna $i(t)$ och $u(t)$ är markerad i kretsen.
Låt $p(t)$ vara effekten levererad av strömkällen.

Bestäm $i(t)$, $u(t)$ och $p(t)$ vid följande tider:

a) Innan brytaren stängs: $t = 0^-$.

b) Direkt efter brytaren stängs: $t = 0^+$.

c) Länge efter brytaren stängs: $t \to \infty$.



Solutions, EI1120 KS2 2016-02-19

1)

At $t < 0$, the step-function makes the only independent source in this circuit have a zero value. We therefore assume[1] that the inductor's current is zero.

When the step-function changes, the voltage-source value becomes U. Then KCL in the top (or bottom) node gives

$$\frac{u(t) - U}{R_1} + gu(t) + i(t) = 0.$$

This is one equation in two unknowns. But the unknowns are the two quantities in the inductor, so we can relate them by the inductor's equation, $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$. We choose to keep the current, $i(t)$, as this is the inductor's continuous variable. That is nicer to work with: we can assume its initial value at $t = 0^+$ is equal to the earlier equilibrium value at $t = 0^-$; and we can differentiate it if we want to find $u(t)$, which is nicer that integrating, as we don't get any integration constant (initial value) to consider.

After eliminating $u(t)$, we have an ODE in $i(t)$,

$$\frac{L\frac{\mathrm{d}i(t)}{\mathrm{d}t}}{R_1} - \frac{U}{R_1} + gL\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t) = 0.$$

Making this neater, the ODE is put into the form $y' + ay = b$,

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{L\left(\frac{1}{R_1} + g\right)}\,i(t) \;=\; \frac{U}{LR_1\,\left(\frac{1}{R_1} + g\right)} \qquad \text{or} \qquad \frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R_1}{L\,(1 + gR_1)}\,i(t) \;=\; \frac{U}{L\,(1 + gR_1)},$$

from which the current is found to have this general solution,

$$i(t) = \frac{U}{R_1} + k\mathrm{e}^{-t/\left(\frac{L}{R_1}(1+gR_1)\right)}$$

where k is to be found from the initial condition. Assuming (as above) that $i(0) = 0$, we have at $t = 0$,

$$i(0) = 0 = \frac{U}{R_1} + k\cdot\mathrm{e}^0 = \frac{U}{R_1} + k \qquad \Longrightarrow \qquad k = -\frac{U}{R_1}.$$

The final expression is then

$$i(t) = \frac{U}{R_1}\left(1 - \mathrm{e}^{-t/\left(\frac{L}{R_1}(1+gR_1)\right)}\right) \qquad (t > 0).$$

Notice that R_2 was irrelevant to the solution, being in series with a current source.

Checks!

It is a 'basic duty' to do a dimensional check on the answer. The U/R_1 term gives a current to match the $i(t)$; good. The term gR_1 is dimensionless, as g relates a current to a voltage, and R_1 relates a voltage to a current; so it is dimensionally correct to add this to 1. The remaining term $t/(L/R_1)$ is time divided by a time-constant, which is also dimensionless, as is required for the argument of the exponential. So this all seems ok.

[1] This isn't absolutely necessary for a circuit with ideal inductors or capacitors, or when there is a dependent source: we could further check that any current in the inductor a long time ago *would* have died away, by analysing the Thevenin resistance of the equivalent of U,R_1,g,R_2 that the inductor sees. In this circuit that turns out to be a positive resistance, so we can expect the current in the initial equilibrium to be zero, zero initial conditions: $i(0^-) = 0$. You are allowed to assume for these exams that the circuit has this property of stablity.

A reasonableness check is also good! After a long time, we expect the inductor to be like a short circuit, so that $u(\infty) = 0$. In this case, $g\,u(t) = 0$, so the current $i(t)$ is just U/R_1. That fits with our solution: when $t \to \infty$, the exponential term disappears, leaving U/R_1.

Another way: reduce circuit to a Thevenin source and the inductor.

By the KCL shown above, a Thevenin equivalent can be found for the circuit (excluding the inductor) that is seen at the inductor's terminals. Instead of eliminating $u(t)$ or $i(t)$ from this KCL equation, we can arrange the equation to show the relation *between* these variables,

$$\frac{u(t) - U}{R_1} + gu(t) + i(t) = 0 \qquad \Longrightarrow \qquad u = \frac{U}{1 + gR_1} - \frac{R_1}{1 + gR_1}\, i.$$

By comparing this to a Thevenin source, where $u = U_{\mathrm{T}} - iR_{\mathrm{T}}$, we find that

$$U_{\mathrm{T}} = \frac{U}{1 + gR_1} \qquad \text{and} \qquad R_{\mathrm{T}} = \frac{1}{\frac{1}{R_1} + g}.$$

From these, the usual solution of the current in a series circuit of U_{T}, R_{T} and L can be made. The final value is the source's short-circuit current, the time-constant is L/R_{T}, and the initial value is found from equilibrium and continuity.

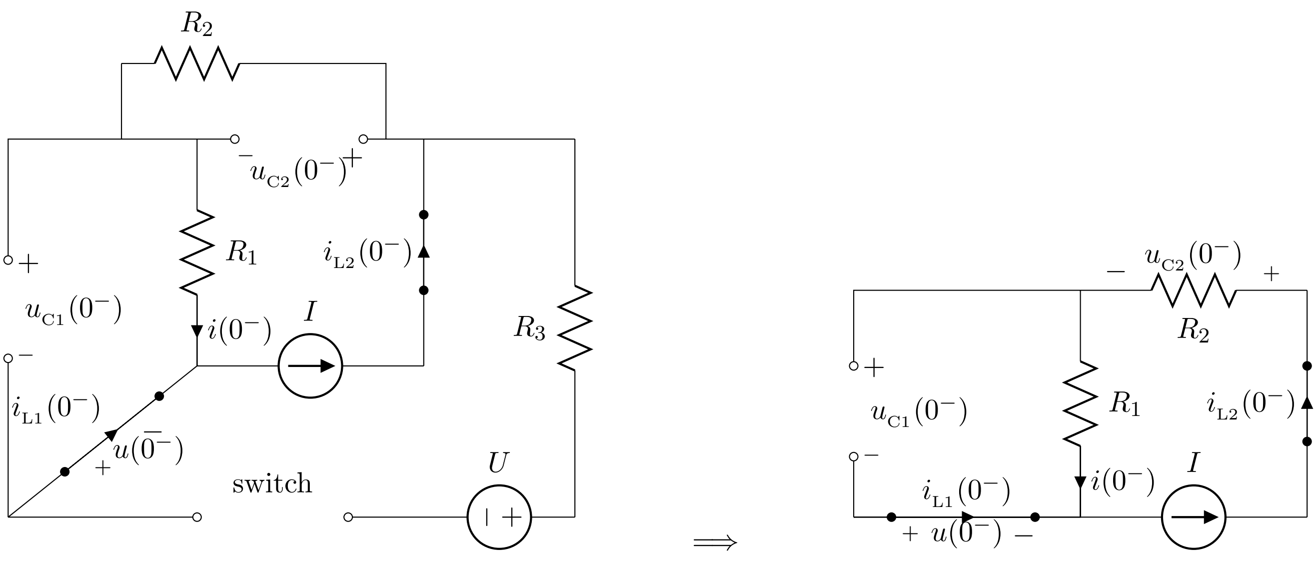
Another way of finding the Thevenin equivalent is by short- and open-circuit calculations.
In short-circuit, $u = 0$, so the dependent current source is zero: thus $i_{\mathrm{sc}} = U/R_1$.
In open-circuit, $i = 0$, KCL gives $(U - u)/R_1 = gu$, leading to $u_{\mathrm{oc}} = \frac{U}{1+gR_1}$.
By the relations $U_{\mathrm{T}} = u_{\mathrm{oc}}$ and $R_{\mathrm{T}} = u_{\mathrm{oc}}/i_{\mathrm{sc}}$, the same result as above is found.

2)

At $t = \{0^-, 0^+, \infty\}$, find $i(t)$ in R_1, $u(t)$ across L_1, and $p(t)$ from the current-source.

a) Equilibrium, *before* the switch closes: $t = 0^-$.

Assuming equilibrium, the capacitors and inductors are replaced by open- and short circuits. The circuit is then solved by dc analysis. The switch prevents the entire outer branch (U, R_3) from being part of the circuit; a simplified form of the relevant circuit is shown on the right. The continuous variables of all inductors and capacitors are included in the diagram, as they need to be found in order to solve the case where $t = 0^+$.



Inspection of these diagrams shows that all the current I flows in a loop around I, L_2, R_2 and R_1.

The requested answers are $i(0^-) = I$, $u(0^-) = 0$, $p(0^-) = I^2(R_1 + R_2)$.

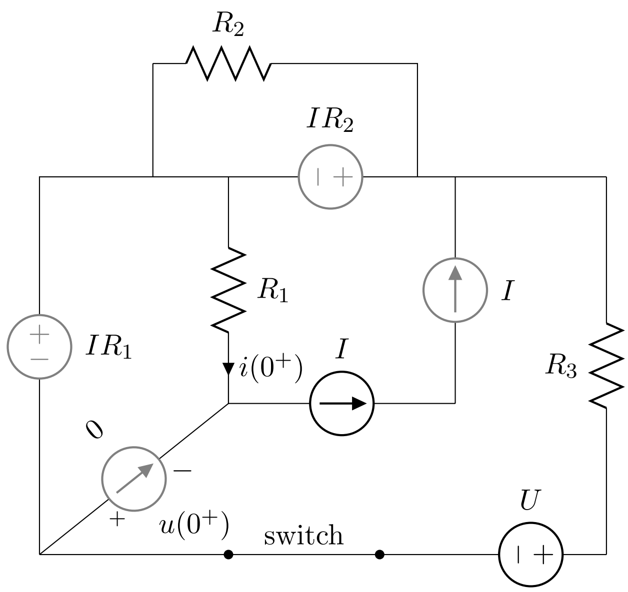
b) Continuity, immediately after the switch closes: $t = 0^+$.

The equilibrium has now been disturbed, but the energy-storing components (inductors and capacitors) will not have changed their energy, and thus their continuous variables, instantaneously.

The first step to solving for $t = 0^+$ is therefore to find what values these continuous variables had at $t = 0^-$, so that we can insert these in the new circuit. From the diagrams in subquestion 'a', we find the following:

$$i_{\mathrm{L1}}(0^-) = 0, \qquad i_{\mathrm{L2}}(0^-) = I, \qquad u_{\mathrm{C1}}(0^-) = IR_1, \qquad u_{\mathrm{C2}}(0^-) = IR_2.$$

Now we include these in the circuit, representing these known quantities as voltage- and current-sources. The extra branch containing the switch also needs to be included now, as the switch is closed.



(It might help to make a further simplification, and to redraw this more clearly. We saw that $i_{\mathrm{L1}}(0^-) = 0$, so the source i_{L1} can be replaced with an open circuit, making the diagram simpler.)

The requested answers are $i(0^+) = I$, $u(0^+) = 0$, $p(0^+) = I^2(R_1 + R_2)$, just as before.[2]

Further explanations:

The current i is found by KCL, bearing in mind $i_{\mathrm{L1}} = 0$.

The voltage u is found by KVL in the loop with IR_1 and R_1.

The power from the source I is found by determining the voltage across the source, which is the sum of IR_2 (on capacitor C_2) and IR_1 (across R_1). *Important point:* why have we ignored the voltage across the "pretend current source I" that models L_2? That sounds dangerous: we know a current source could have a voltage across it. When two similar-looking sources are in series, we've briefly discussed that their currents must be equal, or else they contradict each other and the circuit has no defined solution. Even if they have the same current, we cannot directly calculate the voltage across each, except by appealing to the argument of symmetry, to say that they share the voltage. But this case is special: we know that

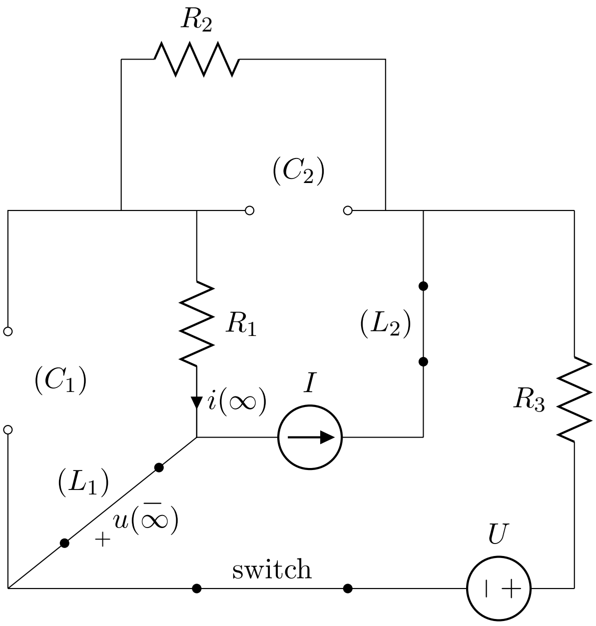
[2] Interesting! It doesn't have to be like that: it just happens to be that these particular quantities are the same at $t = 0^-$ and $t = 0^+$, even though they are not directly the continuous quantities of capacitors or inductors. There *are* some other quantities in this circuit that change when the switch closes. An extra current immediately starts to flow in U and R_3 when the switch closes: by KVL this current must be $\frac{U - u_{\mathrm{C1}}(0^-) - u_{\mathrm{C2}}(0^-)}{R_3}$. The currents in the resistors R_2 and R_1 cannot change immediately: this is dictated by Ohm's law, as the resistors are in parallel with capacitors, whose voltages are continuous. So the extra current passes through the capacitors, whose voltages therefore start changing, i.e. $\mathrm{d}u_{\mathrm{Cx}}/\mathrm{d}t \neq 0$. The voltage $u(t)$ across L_1 then starts to deviate from zero (KVL around R_1, u_{C1}, $u(t)$), so the inductor current can start changing too.

the source representing L_2 must have a *constant* current of I, since it is in series with a genuine current source (that gives I at all times, not just at $t = 0^+$). If the current in that inductor does not change, then $\frac{\mathrm{d}i_{\mathrm{L2}}(t)}{\mathrm{d}t} = 0$, from which the voltage across this inductor must be zero. Tricky.

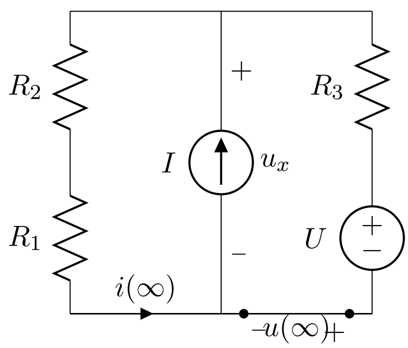
c) Equilibrium, a long time after the switch has closed: $t \to \infty$.

Assuming equilibrium, the capacitors and inductors are replaced by open- and short-circuits respectively. This is rather more complicated than the initial equilibrium, as the switch has introduced a further branch of U and R_3 into the circuit. We try drawing the diagram in a clearer way on the right, in order to find the necessary quantities of u, i, and the voltage across the current source, marked u_x.

In subquestion 'a' we marked further quantities such as u_{C1}, as we needed these for the continuity calculation in 'b'. But here, we only care about finding the requested i, u and p.



$\Longrightarrow$



Clearly, $u(\infty) = 0$. But $i(\infty)$ and $p(\infty)$ both need to be solved in further steps. If we find the unknown u_x, then they can be directly solved from this, as $i(\infty) = \frac{u_x}{R_1+R_2}$, and $p(\infty) = u_x I$.

We could find u_x by KCL (nodal analysis), or superposition, or source-transformation on U and R_3, or doubtless other methods too ...! From the nodal method,

$$u_x = \frac{(R_1 + R_2)\,(U + IR_3)}{R_1 + R_2 + R_3}.$$

The requested answers are then $i(\infty) = \frac{U+IR_3}{R_1+R_2+R_3}$, $u(\infty) = 0$, $p(\infty) = \frac{R_1+R_2}{R_1+R_2+R_3}\left(UI + I^2R_3\right)$.