KTH EI1120 Elkretsanalys (CENMI), Tentamen 2016-03-22 kl 08–13

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ...

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. *Senare deltal brukar vara svårare* att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_k vara poängen man fick från kontrollskrivning KS1, och b_k poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a,a_\mathrm{k})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b,b_\mathrm{k})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a,a_\mathrm{k}) + \max(b,b_\mathrm{k}) + c + h}{A+B+C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Nathaniel Taylor (073 919 5883)

Del A. Likström

1) [4p] Bestäm effekterna som försörjs från kretsen till de följande komponenterna:

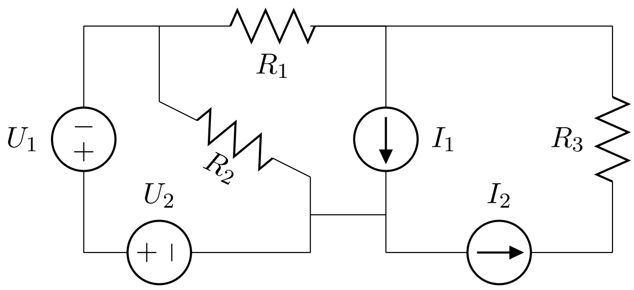
a) [1p] motståndet R_3

b) [$\frac{1}{2}$p] motståndet R_2

c) [$\frac{1}{2}$p] motståndet R_1

d) [1p] spänningskällan U_2

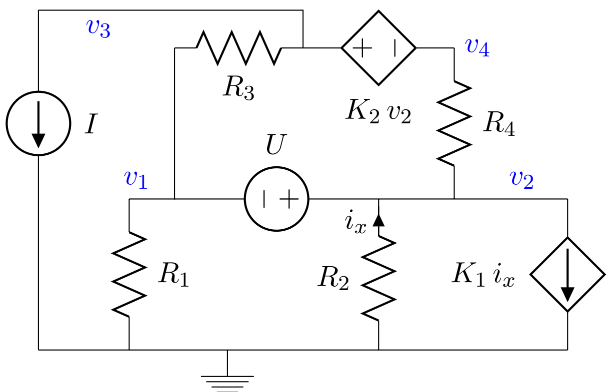
e) [1p] strömkällan I_2



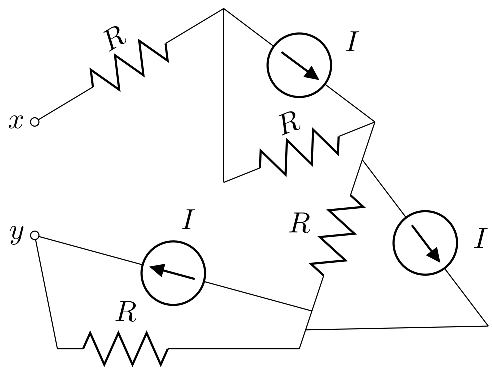
2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1, v_2, v_3, v_4.

Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.



3) [4p]

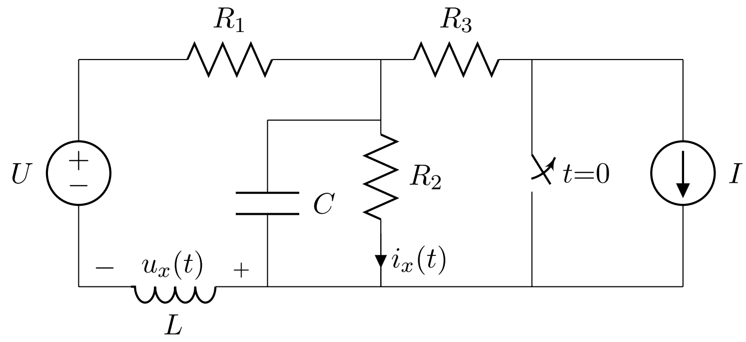


a) [3p] Bestäm Nortonekvivalenten av kretsen, med avseende på polerna 'x' och 'y'. Rita upp ekvivalenten inklusive polerna.

b) [1p] En spänningskälla U ansluts mellan kretsens poler x-y, med sin '+' pol till kretsens pol y. Vilket värde måste spänningen U ha, uttryckt som en funktion av I och R, för att den maximala möjliga effekten ska absorberas av källan från kretsen?

Del B. Transient

4) [5p] Bestäm följande:

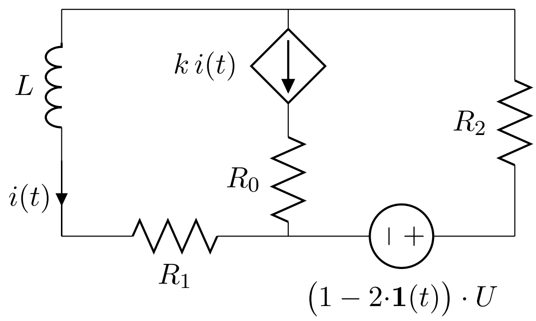


a) [2p] Vid $t = 0^-$,
effekten levererad av strömkällan I,
strömmen i_x.

b) [2p] Vid $t = 0^+$,
strömmen i_x,
spänningen u_x över spolen.

c) [1p] Vid $t \to \infty$,
energin lagrad i kondensatorn C,
effekten levererad av strömkällan I.

5) [5p]

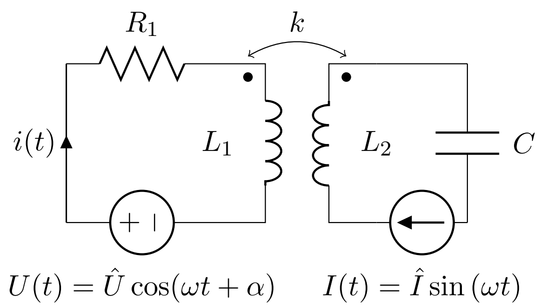


Bestäm $i(t)$, för $t > 0$.

Obs: $\mathbf{1}(t)$ är enhetsstegfunktionen.

Del C. Växelström

6) [5p]



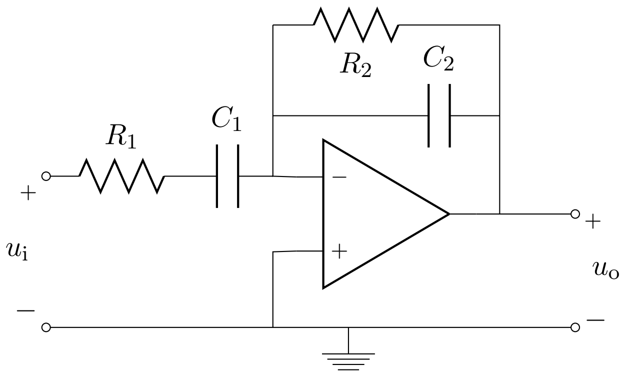
a) [3p] Bestäm $i(t)$ med villkoret $\hat{I} = 0$.
Tips: i så fall kan en mycket förenklad krets analyseras.

b) [2p] Bestäm $i(t)$ *utan* villkoret ovan.
Båda källor betraktas nu som aktiva.

7) [5p]

a) [2p] Bestäm kretsens nätverkfunktion,

$$H(\omega) = \frac{u_\mathrm{o}(\omega)}{u_\mathrm{i}(\omega)}.$$



b) [1p] Visa att svaret till deltal 'a' kan skrivas i den följande formen,

$$H(\omega) = \frac{-\mathrm{j}\omega/\omega_1}{(1+\mathrm{j}\omega/\omega_2)(1+\mathrm{j}\omega/\omega_3)}.$$

c) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'. Anta att $100\,\omega_1 = \omega_2$, och att $\omega_2 \ll \omega_3$. Markera viktiga punkter och lutningar.

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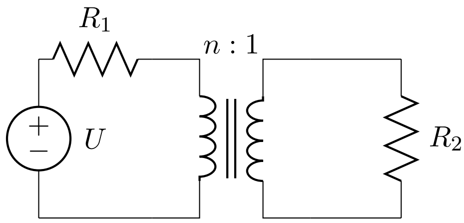
8) [3p]

Värmeelementen R_2 matas genom en lång, tunn ledning R_1 från källan U; dessa parametrar är fastställda. Bestäm kvoten n av transformatorn, för att värmeelementen ska utveckla så mycket värme som möjligt.



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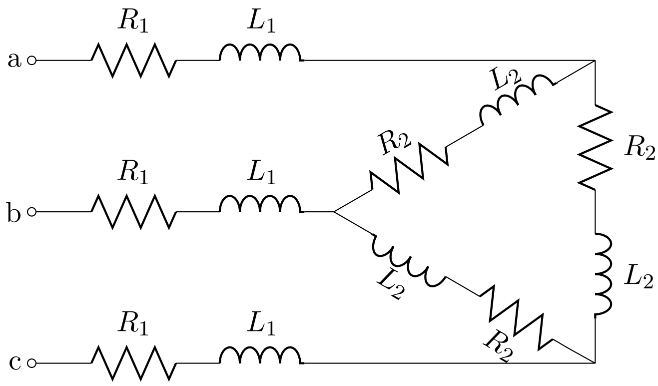
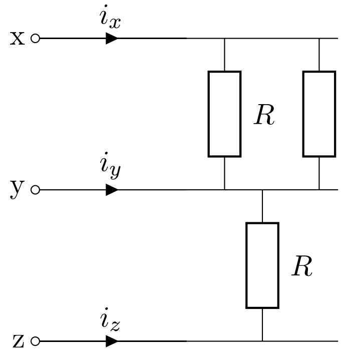
9) [5p]

Polerna a,b,c och x,y,z i kretsarna nedan visar anslutningar till trefas spänningskällor. Båda källor har vinkelfrekvens ω och huvudspänning U (därför blir t.ex. $|u_\mathrm{ab}| = U$ och $|u_\mathrm{xy}| = U$).
Som vanligt kan antas: källorna är ideala och balanserade, och effektivvärdeskala används.





a) [2p] Kretsen till vänster modellerar en Δ-last (R_2, L_2) kopplad till källan genom en kabel som har motstånd R_1 och induktans L_1. Vilken effektfaktor (PF) matar källan, vid polera a,b,c?

b) [2p] Tre kondensatorer i Δ anslutning ska kopplas till källans poler i den vänstra kretsen, för att helt kompensera lasten och kabeln, så källan matar PF $= 1$. Vilket värde C måste varje kondensator ha?

c) [1p] Kretsen till höger visar tre motstånd R. De skulle kunna kopplas som en balanserad trefas last, men de har felkopplats på obalanserat sätt. Bestäm magnituden av den största av de markerade linjeströmmarna (i_x, i_y eller i_z).

Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1120, VT16, 2016-03-22)

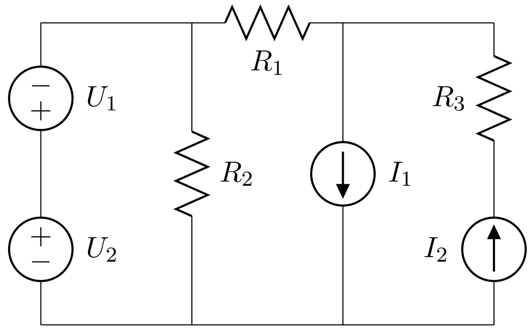
Q1

Some re-drawing may be helpful. However – as usual – it is not obvious whether everyone will find it worth taking the extra time to do this, particularly when it involves the risk of copying something wrongly into the new diagram!

The powers delivered to (equivalently: 'absorbed by') the listed components are found by the usual methods.

For a resistor R it is often easiest to find the magnitude of voltage u across it, *or* current i through it, then to use the relation $P = i^2R$ or $P = u^2/R$.

For a source, find the quantity that the source *doesn't* determine (a voltage-sources's current or a current-source's voltage) then multiply this with the source value, taking care about the relative directions of current and voltage definition.



a) $P_{\text{R3}} = I_2^2 R_3$.
This resistor R_3 is series-connected to current-source I_2, which therefore determines its current.

b) $P_{\text{R2}} = \frac{(U_1-U_2)^2}{R_2}$ note that $(U_1 - U_2)^2 = (U_2 - U_1)^2$.
This resistor R_2 is parallel-connected to a voltage that is the difference between the source-voltages U_1 and U_2. If you don't like this sort of argument based on "parallel-connected" and "rest of circuit is irrelevant" and so on, then consider KVL around U_1, U_2 and R_2.

c) $P_{\text{R1}} = (I_1 - I_2)^2 R_1$.
KCL above source I_1 gives a current $I_1 - I_2$ flowing left to right through R_1.
Only the magnitude of this current matters, since $(I_1 - I_2)^2 = (I_2 - I_1)^2$.

d) $P_{\text{U2}} = U_2 \left(\frac{U_1-U_2}{R_2} + I_2 - I_1 \right)$.
The current in source U_2 can be found by KCL in three branches: R_2, I_1 and I_2. This is probably most obvious if KCL is done in the central node (to the right of U_2). By force of habit, I've written KCL to give the current out of the + terminal of U_2; multiplied by U_2 this would give the power *supplied* by U_2 instead of the requested power *absorbed*, so an extra negative sign is needed in order to describe the power delivered *to* the source.

$$P_{\text{absorbed}} = -U_2 \left(\frac{U_2 - U_1}{R_2} + I_1 - I_2 \right) = U_2 \left(\frac{U_1 - U_2}{R_2} + I_2 - I_1 \right).$$

e) $P_{\text{I2}} = I_2 \big(U_1 - U_2 + I_1 R_1 - I_2(R_1 + R_3) \big)$.
Now we must find the voltage across source I_2. Let's define this voltage as u_{I2}, with the +-reference at the bottom of the source (where the current-arrow starts) so that $u_{\text{I2}} I_2$ will be the power input to the source. The most obvious loop for KVL is the entire outer loop of the circuit, which avoids other current sources (unknown voltages). The voltage sources in this loop have voltages that are immediately known, and the voltages across the two resistors are easily found by Ohm's law as the currents in these resistors are set by the current sources and KCL as already noted in the earlier solutions. The complete KVL is

$$0 = U_2 - U_1 - (I_1 - I_2)R_1 + I_2 R_3 + u_{\text{I2}},$$

yielding u_{I2}, from which the source's absorbed power can be found as described above,

$$u_{\text{I2}} = U_1 - U_2 + R_1 I_1 - (R_1 + R_3) I_2.$$

Q2

Extended nodal analysis ("the simple way")

Let's define the unknown currents in the voltage sources, with the positive direction going into the source's + terminal: i_α in the independent voltage source U, and i_β in the dependent voltage source.

Write KCL (let's take outgoing currents) at all nodes except ground:

$$\begin{aligned}
\text{KCL}(1): \quad 0 &= \frac{v_1}{R_1} + \frac{v_1 - v_3}{R_3} - i_\alpha & (1)\\
\text{KCL}(2): \quad 0 &= \frac{v_2}{R_2} + \frac{v_2 - v_4}{R_4} + i_\alpha + K_1 i_x & (2)\\
\text{KCL}(3): \quad 0 &= I + \frac{v_3 - v_1}{R_3} + i_\beta & (3)\\
\text{KCL}(4): \quad 0 &= \frac{v_4 - v_2}{R_4} - i_\beta. & (4)
\end{aligned}$$

The voltage sources introduced the problem of two extra unknowns in the above equations; they can solve this problem by providing two extra equations without further unknowns:

$$\begin{aligned}
v_2 - v_1 &= U & (5)\\
v_3 - v_4 &= K_2\, v_2. & (6)
\end{aligned}$$

The controlling variables of the dependent sources need to be defined in terms of the other known or unknown quantities. Our dependent voltage source's controlling variable is the potential v_2, which is an unknown that we already introduced in the KCL equations: nothing more needs to be done for that. Our dependent current source's controlling variable is a current i_x marked in R_2. This can be described as

$$i_x = -\frac{v_2}{R_2}. \quad (7)$$

$$(8)$$

The above is a sufficient set of equations for a solution.

Some double-checking is shown below, comparing a circuit simulator's output with a solution of the above equations.

```
% These equations can be solved in (e.g.) Matlab symbolic toolbox,
s = solve ( ...
  '0 = v1/R1 + (v1-v3)/R3 - ia', ...
  '0 = v2/R2 + (v2-v4)/R4 + ia + K1*ix', ...
  '0 = I_ + (v3-v1)/R3 + ib', ...
  '0 = (v4-v2)/R4 - ib', ...
  'v2 - v1 = U', ...
  'v3 - v4 = K2*v2', ...
  'ix = -v2/R2', ...
  'v1, v2, v3, v4, ix, ia, ib' )

% the symbolic results [rather long!] can be shown by this:
%for f=fields(s)', disp(f{1}); disp( s.(f{1}) ); disp(''); end

% now set some [arbitrary] numeric values, and substitute them
U=20,  I_=0.1,   K1=3,  K2=0.5,  R1=14,  R2=68,  R3=5,  R4=9
for f=fields(s)', fprintf(' %s: %f\n', f{1}, double(subs(s.(f{1}))) ); end
  ia: -1.663571
  ib: -2.593571
  ix: -0.465000
  v1: 11.620000
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  v2: 31.620000
  v3: 24.087857
  v4:  8.277857

% For comparison, the following input into the SPICE program
% specifies the same circuit, with the numeric values as used
% in the above solution in Matlab,
EI1120_VT16_TEN
V1    2 1  DC  20.0
I1    3 0  DC   0.1
R1    1 0      14.0
R2    2 5      68.0
R3    3 1       5.0
R4    4 2       9.0
E1    3 4  2 0  0.5
V0    0 5  DC   0
F1    2 0  V0   3.0
.OP
.PRINT DC V(1) V(2) V(3) V(4)
.END

% and this results in the following solution of potentials etc,:
     POTENTIALS
        v1:  11.6200
        v2:  31.6200
        v3:  24.0879
        v4:   8.2779
     VSRC
         i:  -1.664      ('ia')
     VCVS
         v:  15.810
         i:  -2.59       ('ib')
     CCCS
         i:  -1.39       ('K1 ix')
```
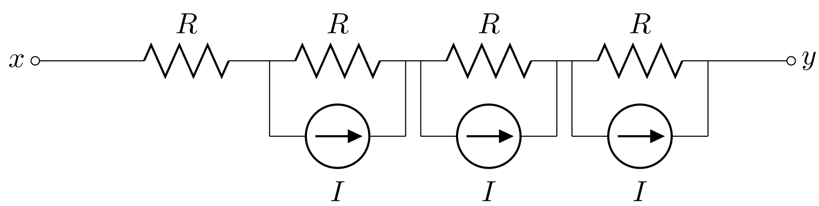

Q3

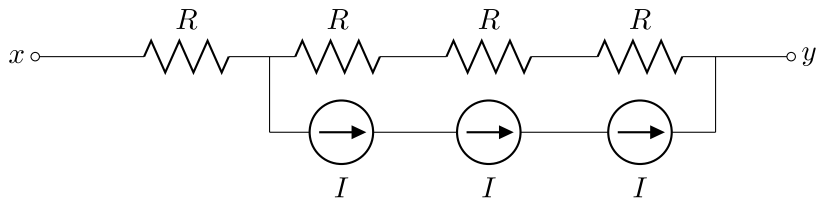
It might help to draw this same circuit in a neater form. The following diagram is exactly the same as the original one in the question.



a) Find the Norton equivalent between terminals x and y: $I_{\mathrm{N}} = \frac{3}{4}I$ and $R_{\mathrm{N}} = 4R$.

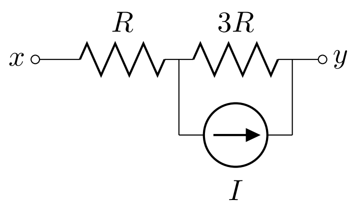
One method: keeping the current sources

The following diagram is *not* exactly the same as the previous one, but it has the same behaviour at the terminals x-y, which is what we're ultimately interested in. The difference is that the nodes where two resistors and current sources join have each been split into two separate nodes. We can justify this step by noting that the nodes between the resistors would not be affected by being connected to the current sources, since in each case there is one current source putting current in, and another current source taking equal current out. This conversion sometimes goes under the name of Blakesley's current-source shift theorem. Alternatively, we can appeal to symmetry: the same voltage is expected across each identical series-connected component, so each separated pair of nodes still has the same potential, so joining them would make no difference.



By simplifying each series set of identical components, the circuit can be reduced to a simpler form as shown on the right.

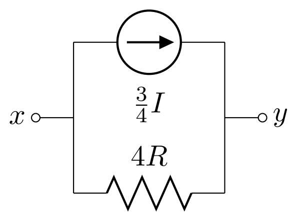
The short-circuit current (from y to x) of this circuit is the source current of its Norton equivalent. It can be found by current division:



$$i_{\text{sc}} = I_{\text{N}} = I\frac{3R}{R+3R} = \frac{3}{4}I.$$

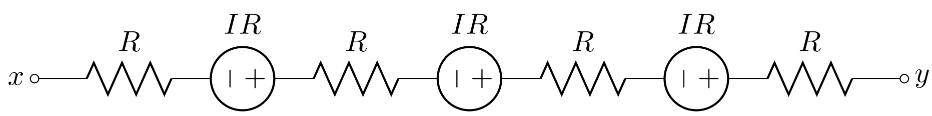
The source resistance can be found by zeroing the current source (open-circuit) and finding the equivalent resistance, which is the circuit's Norton or Thevenin resistance: $R_{\text{N}} = 4R$.

From the above I_{N} and R_{N}, the Norton equivalent can be drawn, taking care that the direction of the source with respect to the marked terminals x and y gives the right direction of current.

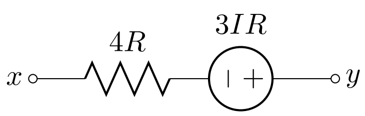


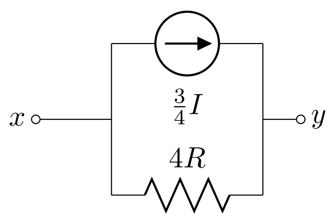
Another method: source-transformations

By source-transforming each parallel I-R pair in the original circuit, a single series-connected branch can be made, that has identical terminal-properties to the original circuit.



This easily simplifies to a single Thevenin source (left), or to its Norton equivalent (right) by a further source-transformation.





Retrospectively, this seems easier than the first method. Because we were looking for a Norton equivalent, it seemed wise to keep the current sources and try to simplify them. However, this second method rapidly simplified into a Thevenin equivalent, which was easily converted to a Norton. (It also has the advantage of not needing to justify the step of disconnecting the current-source nodes from the resistor nodes.)

b) What value U should a voltage source have if connected y-x (with + terminal connected to y) and wanting to extract maximum power from the circuit?

Maximum power output from the circuit is obtained when it is loaded to half of its short-circuit current $i_{\rm sc}/2$, or equivalently to half of its open-circuit voltage $u_{\rm oc}/2$, or by a resistance equal to its source-resistance!

The source is connected so that U is in the same direction as the open-circuit voltage $3IR$.
We need then $U = \frac{u_{\rm oc}}{2} = \frac{3}{2}IR$ to obtain maximum power into this source.

Q4

a) At $t = 0^-$, equilibrium with switch closed.

Power out of current-source I: $P_{\rm I}(0^-) = 0$.
The switch short-circuits the current source, so the voltage across this source is fixed to zero. Therefore, no energy is given to or taken from the charge that passes through the source.

Marked current i_x: $i_x(0^-) = \frac{UR_3}{R_1R_2+R_2R_3+R_3R_1}$.
The equivalent resistance of R_1, R_2 and R_3 is $R_{\rm eq} = R_1 + \frac{R_2R_3}{R_2+R_3}$.
The current out of the voltage source is found from $U/R_{\rm eq}$.
This current divides between R_2 and R_3, so i_x (in R_2) is the fraction $\frac{R_3}{R_2+R_3}$ of the total:

$$i_x = \frac{U}{R_1 + \frac{R_2R_3}{R_2+R_3}} \cdot \frac{R_3}{R_2+R_3} = \frac{UR_3}{R_1R_2+R_2R_3+R_3R_1}.$$

b) At $t = 0^+$, switch newly opened.

Marked current i_x: $i_x(0^+) = \frac{UR_3}{R_1R_2+R_2R_3+R_3R_1}$.
This is the same as $i_x(0^-)$! That's because the capacitor's voltage is a continuous variable, so it hasn't changed instantaneously when the switch was opened, and the current i_x is in a resistor that is connected in parallel with the capacitor and thus has the same voltage.

Marked voltage u_x, across the inductor: $u_x(0^+) = 0$.
The left branch (L,U,R_1) is connected in parallel with the capacitor C. This capacitor's voltage is continuous, so the left branch is connected to this same voltage at $t = 0^-$ and at $t = 0^+$, regardless of the switch in the right part of the circuit having changed. There is therefore no reason for anything in the left branch to change: the inductor's voltage remains at the zero value we expect for an inductor's voltage in equilibrium.
If you want to do this more formally, then write the KVL equation for the loop of the left branch and the capacitor: the capacitor's voltage is i_xR_2, and the inductor's current is $U/R_{\rm eq}$ (see subquestion 'a'), which passes through R_1. Solve this for u_x.
It might sound strange that nothing changes. Remember that this is only *immediately after* the switch opens. The opening switch causes the current through R_3 to change (unless it just happened that I was chosen to be the same as the initial current in R_3!); KCL above R_2 requires this change of current to flow in the capacitor, which will *start* changing its voltage. So gradually, the other quantities such as i_x, u_x, etc, can change too.

c) As $t \to \infty$, equilibrium with switch open.

For both of these questions it is helpful to calculate the voltage across the capacitor. It's an equilibrium situation again, so the capacitor is open-circuit and the inductor is short-circuit.

Let's define the node below R_2 as the earth node, and the node above R_2 as potential v. By nodal analysis on three branches,

$$\frac{v-U}{R_1} + \frac{v}{R_2} + I = 0 \quad\Longrightarrow\quad v = \frac{(U-IR_1)R_2}{R_1+R_2}.$$

The stored energy in the capacitor is $\frac{1}{2}Cv^2$, which is $E_{\rm C} = \frac{1}{2}C\left(\frac{(U-IR_1)R_2}{R_1+R_2}\right)^2$.

The power out of current-source is $(-v+IR_3)I$, which is $P_{\rm I} = \left(\frac{(IR_1-U)R_2}{R_1+R_2} + R_3I\right)I$.
To show this, the voltage across the current source is found: let's call this u, with its positive reference at the bottom of the source so that uI is the power delivered by the source.
By KVL, this voltage is $u = -v + IR_3$.
If instead we define this voltage with positive reference at the top of the current source, then the value of u is negated, but we also need to negate the expression uI (as it then gives the power *in* to the source), so the final answer becomes the same.

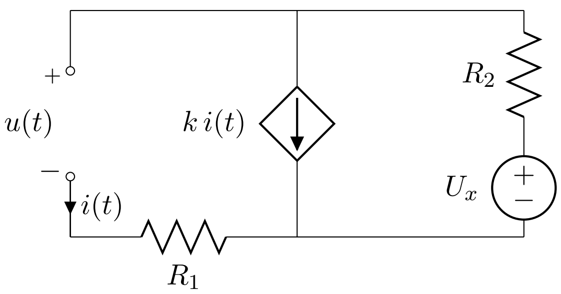
Q5

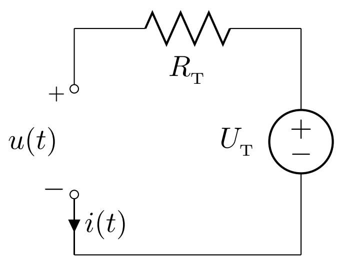
Here we are looking for the current marked in the inductor: $i(t) = \dfrac{U}{R_1+(1{+}k)R_2}\left(2\,\mathrm{e}^{-\frac{R_1+(1+k)R_2}{L}t} - 1\right)$.

Two approaches will be shown: each could in turn be split into several different choices.

By two-terminal equivalent and initial/final values.

Removing the inductor, a 'dc' circuit (static: no 'state' that remembers the past) remains. This can be converted to a Thevenin (or Norton) equivalent. The diagrams below show the original circuit (left) and equivalent (right). When drawing the original circuit, the irrelevant resistor R_0 has been omitted, and the voltage source (which actually changes its value at $t=0$) has been replaced with a source U_x. The Thevenin equivalent of the circuit either at $t>0$ or $t<0$ can then be found by setting U_x to the appropriate value.





The circuit has three parallel branches. The currents in two of them are already marked as i and ki. The current in R_2 can be found by KVL around the outer loop. KCL can then be written for the three currents,

$$i + ki + \frac{u + iR_1 - U_x}{R_2} = 0,$$

from which the relation between the two unknowns u and i can be found, and compared to the relation for a Thevenin source,

$$u = U_x - (R_1 + (1{+}k)R_2)\,i \qquad \text{c.f.} \quad u = U_{\rm T} - R_{\rm T}i$$

which shows the Thevenin voltage to be equal to the voltage-source U_x, and the Thevenin resistance to be $R_1 + (1{+}k)R_2$.

At $t<0$, the step function $(1 - 2\cdot\mathbf{1}(t))\,U$ in the original circuit's voltage source had a value of U. The Thevenin equivalent of the circuit for $t<0$ is therefore with $U_x = U$. An inductor connected to this equivalent should have the same current and voltage as in the original circuit.

In equilibrium an inductor is like a short-circuit, so the equilibrium current $i(0^-)$ is the same as the short-circuit current. By continuity this is also the inductor's current just after the step change.

$$i(0^+) = i(0^-) = \frac{U_{\rm T}}{R_{\rm T}} = \frac{U}{R_1+(1{+}k)R_2}.$$

After the step, $t > 0$, the voltage source's value becomes $-U$, so the Thevenin equivalent is with $U_x = -U$. The equilibrium value of the inductor's current is then

$$i(\infty) = \frac{U_{\mathrm{T}}}{R_{\mathrm{T}}} = \frac{-U}{R_1 + (1{+}k)R_2}.$$

The time-constant is

$$\tau = \frac{L}{R_{\mathrm{T}}} = \frac{L}{R_1 + (1{+}k)R_2}.$$

By the relation $i(t) = i(\infty) + (i(0) - i(\infty))\,\mathrm{e}^{-t/\tau}$, the time-function is found as

$$i(t) = \frac{U}{R_1 + (1{+}k)R_2} \left(2\,\mathrm{e}^{-\frac{R_1+(1+k)R_2}{L}t} - 1\right) \qquad (t \geq 0)$$

The short-circuit and open-circuit method could alternatively have been used to find the Thevenin equivalent, but this is probably more work than just writing the expression relating i and u.

By direct differential-equation solution.

The circuit has three parallel branches. The inductor's current $i(t)$ is already marked; let's define its voltage as $u(t)$ with positive reference at the upper side. At $t > 0$, the voltage source's value is $-U$. By KCL,

$$i(t) + ki(t) + \frac{u(t) + R_1 i(t) - (-U)}{R_2} = 0.$$

This expression used KVL around the outer loop to find the current in the resistor R_2.

Now substitute the relation between $u(t)$ and $i(t)$ given by the equation of an inductor: $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, so that there's only one unknown in the resulting differential equation. (We'd like to eliminate $u(t)$ instead of $i(t)$ because it's not the continuous variable, so it's not so directly obvious how to handle the initial condition.)

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R_1 + (1+k)R_2}{L}\,i(t) = \frac{-U}{L}$$

This has a general solution of

$$i(t) = \frac{-U}{R_1 + (1+k)R_2} + A\mathrm{e}^{-\frac{R_1+(1+k)R_2}{L}t},$$

where A must be found from knowledge of the solution at a particular time: in our case, the initial value $i(0)$ can be found by equilibrium and continuity.

This is not trivial. The equilibrium at $t = 0^-$ allows the current to be found by assuming the inductor to have zero voltage due to $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 0$; continuity then equates this to the current at $t = 0^+$.

Using the KCL equation above, with $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 0$ and a source-voltage $+U$ instead of $-U$ (because of the source having a different value before $t = 0$), we find the initial current is

$$i(0^+) = i(0^-) = \frac{U}{R_1 + (1+k)R_2},$$

from which A is found in order to make

$$i(0^+) = \frac{U}{R_1 + (1+k)R_2} = \frac{-U}{R_1 + (1+k)R_2} + A\mathrm{e}^0,$$

which requires

$$A = 2\frac{U}{R_1 + (1+k)R_2},$$

leading to the solution that has already been shown.

Q6

Here, two inductors have mutual coupling.
The coupling coefficient is shown as k, from which mutual inductance can be found as $M = k\sqrt{L_1 L_2}$.

a) Find $i(t)$ given the simplification that $\hat{I} = 0$.

The zero current source means that no current can flow in the loop on the right-hand side. The voltage across L_1 is therefore $L_1 \frac{\mathrm{d}i(t)}{\mathrm{d}t} + 0M$. This means that the mutual inductance can be ignored when calculating what happens in the loop at the left-hand side: the inductor L_1 behaves like a plain "self-inductor".

The voltage source can be represented as a phasor,

$$U(\omega) = \hat{U}\underline{/\alpha},$$

using a cosine as the angle reference, and peak value as the phasor magnitude.

By KVL, $U(\omega) = R_1 i(\omega) + \mathrm{j}\omega L_1 i(\omega)$.

For people who like rectangular form and conjugates,

$$i(\omega) = \frac{\hat{U}\underline{/\alpha}}{R_1 + \mathrm{j}\omega L_1} = \frac{\hat{U}\underline{/\alpha} \cdot (R_1 - \mathrm{j}\omega L_1)}{R_1^2 + \omega^2 L_1^2}.$$

And for people who like polar form,

$$i(\omega) = \frac{\hat{U}}{\sqrt{R_1^2 + \omega^2 L_1^2}} \underline{/\alpha - \tan^{-1} \tfrac{\omega L_1}{R_1}}.$$

Translated back to a time-function (careful to keep the same choice of peak value and cosine-reference),

$$i(t) = \frac{\hat{U}}{\sqrt{R_1^2 + \omega^2 L_1^2}} \cos\left(\omega t + \alpha - \tan^{-1} \frac{\omega L_1}{R_1}\right).$$

b) Find $i(t)$ for the general case where $\hat{I} \neq 0$.

We're still quite lucky.[1] The current source forces a current in the right-hand side, independently of what happens at the left-hand side. Therefore, calculation of the current $i(t)$ at the left hand side can be done as it was in subquestion 'a', but with one extra voltage source representing the voltage $M \frac{\mathrm{d}i_2(t)}{\mathrm{d}t}$ that is induced in L_1 by the current in L_2.

Now convert the sinusoidal time-functions to phasors. Let's represent the voltage source in the same way as before. The current source has the same frequency, so we can find a phasor for this using the same reference (cosine, peak value) and calculate using both together.

$$U(\omega) = \hat{U}\underline{/\alpha}, \qquad \text{and} \qquad I(\omega) = \hat{I}\underline{/\tfrac{-\pi}{2}} = -\mathrm{j}\hat{I}.$$

Taking care about the directions and the dots, KVL in the left-hand side loop gives

$$U(\omega) \;=\; R_1 i(\omega) + \mathrm{j}\omega L_1 i(\omega) + \mathrm{j}\omega M\left(-I(\omega)\right) \;=\; i(\omega)\left(R_1 + \mathrm{j}\omega L_1\right) + \mathrm{j}\omega M(-(-\mathrm{j}\hat{I})),$$

into which the expression for M is substituted, and the multiple negatives and imaginary units are simplified,

$$\hat{U}\underline{/\alpha} \;=\; i(\omega)\left(R_1 + \mathrm{j}\omega L_1\right) - \omega k \sqrt{L_1 L_2}\hat{I}.$$

[1] Lucky?? Well, ... if the right-hand side had its current source replaced by a voltage source, it would have been harder, as the current at the right-hand side would depend on the circuit at the left-hand side.

The sought current, as a phasor, is then

$$i(\omega) \;=\; \frac{\hat{U}\angle\alpha + \omega k\sqrt{L_1L_2}\hat{I}}{R_1 + \mathrm{j}\omega L_1} \;=\; \frac{\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I} + \mathrm{j}\hat{U}\sin\alpha}{R_1 + \mathrm{j}\omega L_1}.$$

We need to get this into polar form in order to write the time-function.

$$i(\omega) \;=\; \sqrt{\frac{\left(\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I}\right)^2 + \left(\hat{U}\sin\alpha\right)^2}{R_1^2 + \omega^2 L_1^2}} \;\angle\, \tan^{-1}\frac{\hat{U}\sin\alpha}{\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I}} - \tan^{-1}\frac{\omega L_1}{R_1}$$

(We'd need to add an extra π to the angle if α is such that $\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I} < 0$. Well done if you considered this. We won't take off points for ignoring it in this particular case.)

The time-function is hardly necessary to write, since it comes directly from the above.
But let's copy-and-paste it anyway, for completeness:

$$i(t) \;=\; \sqrt{\frac{\left(\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I}\right)^2 + \left(\hat{U}\sin\alpha\right)^2}{R_1^2 + \omega^2 L_1^2}} \;\cos\left(\omega t + \tan^{-1}\frac{\hat{U}\sin\alpha}{\hat{U}\cos\alpha + \omega k\sqrt{L_1L_2}\hat{I}} - \tan^{-1}\frac{\omega L_1}{R_1}\right)$$

Q7

a) Find the network function $H(\omega) = \frac{u_\mathrm{o}(\omega)}{u_\mathrm{i}(\omega)}$.

This is a classic configuration of an inverting amplifier.
If we call the input and feedback impedances Z_i and Z_f, then $H = \frac{-Z_\mathrm{f}}{Z_\mathrm{i}}$.

In our case, $Z_\mathrm{i} = R_1 + \frac{1}{\mathrm{j}\omega C_1}$ (series), and $Z_\mathrm{f} = \frac{R_2\frac{1}{\mathrm{j}\omega C_2}}{R_2 + \frac{1}{\mathrm{j}\omega C_2}}$ (parallel).

From the above expressions, or from direct application of KCL at the inverting input, we find

$$H(\omega) = \frac{-\frac{R_2\frac{1}{\mathrm{j}\omega C_2}}{R_2 + \frac{1}{\mathrm{j}\omega C_2}}}{R_1 + \frac{1}{\mathrm{j}\omega C_1}} = \frac{-R_2\frac{1}{\mathrm{j}\omega C_2}}{\left(R_1 + \frac{1}{\mathrm{j}\omega C_1}\right)\left(R_2 + \frac{1}{\mathrm{j}\omega C_2}\right)} = \frac{-\mathrm{j}\omega C_1 R_2}{(1 + \mathrm{j}\omega C_1 R_1)(1 + \mathrm{j}\omega C_2 R_2)}.$$

b) Show that the function from subquestion 'a' can be written as $H(\omega) = \frac{-\mathrm{j}\omega/\omega_1}{(1+\mathrm{j}\omega/\omega_2)(1+\mathrm{j}\omega/\omega_3)}$.

It's convenient to show this by expressing each of the new parameters ($\omega_{1,2,3}$) in terms of the given values of subquestion 'a'. By setting

$$\omega_1 = \frac{1}{C_1R_2}, \qquad \omega_{2|3} = \frac{1}{C_1R_1}, \qquad \omega_{3|2} = \frac{1}{C_2R_2},$$

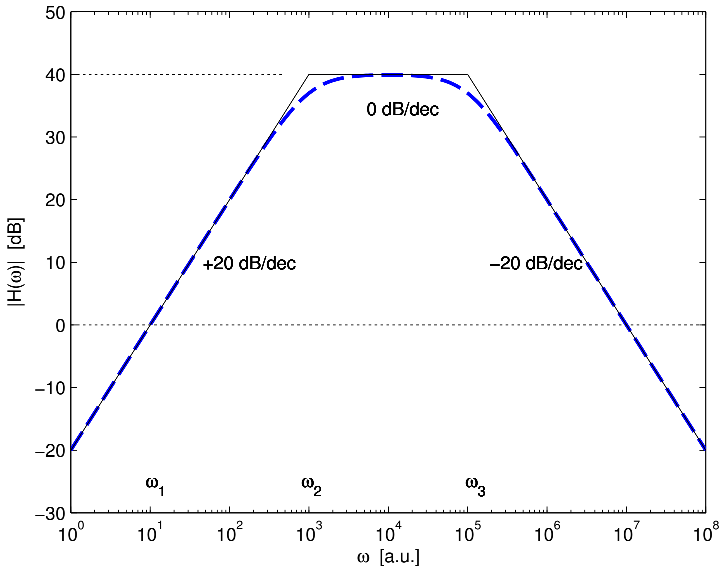
the desired form of $H(\omega)$ becomes equivalent to the solution of subquestion 'a'.
There is nothing to say whether C_1R_1 should correspond to ω_2 or ω_3: the '$_{2|3}$' subscript indicates that either choice could be used, as long as the opposite choice is made for $\omega_{3|2}$.

c) Sketch a Bode amplitude plot of $H(\omega)$. Assume $100\,\omega_1 = \omega_2$, and $\omega_2 \ll \omega_3$.

This is shown on the right: the frequency is in arbitrary units, and the ratio of ω_3/ω_2 has been chosen as 100.

The main features that should be marked are the 40 dB level of the pass-band, the intercept of 0 dB at $\omega = \omega_1$, the the changes of gradient at ω_2 and ω_3, and the gradients of ± 20 dB/decade.

Including the 0 dB/decade gradient in the pass-band is nice but not necessary, as it's obvious for a flat line! The classic asymptotic Bode amplitude plot has just the straight lines; the further curve shows the actual function plotted numerically.



The negative sign in $H(\omega)$ arises because of this being an inverting amplifier. It has no effect on the amplitude plot, as it changes the function's phase, not its magnitude: $|H(\omega)| \equiv |-H(\omega)|$.

Q8

The transformer ratio to maximise power transfer to the load resistor R_2 is $n = \sqrt{R_1/R_2}$.

The transformer has the same power going in as coming out (it is an ideal transformer), so maximising the power in to the left side of the transformer is equivalent to maximising the power in to the resistance R_2.

The equivalent resistance that the Thevenin source (U, R_1) sees at the transformer's input is $n^2 R_2$. For maximum power transfer from the Thevenin source, this needs to be made equal to the Thevenin resistance:

$$n^2 R_2 = R_1 \qquad \Longrightarrow \qquad n = \sqrt{\frac{R_1}{R_2}}.$$

Punctilious people properly perceive the point that this solution could have a positive or negative value: the positive one is intended.[2]

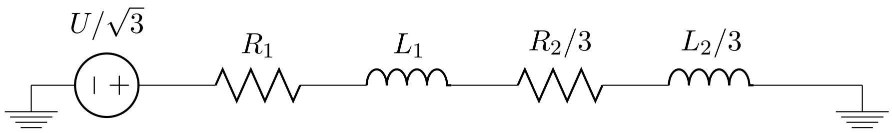
We can assume this is an ac source, as the question comes in the Växelström part of the exam. However, out of kindness there are no inductances or capacitances shown, so the maximum power condition is just as simple as in a dc circuit. Some would argue that the presence of a transformer shows it's an ac situation. That's true in the practical sense: the voltage needn't be sinusoidal ac, but it should change sign frequently. A transformer won't play happily with a voltage source that has a significantly non-zero mean: the flux in the core would need to keep changing in the same direction in order to oppose the applied voltage; in practice, the resistance of the coil and other wires would end up limiting the current. On the other hand, we tend not to worry about practical things in this course … an *ideal* transformer works fine in a dc system, where its "infinite inductance" avoids the build-up of large fluxes … but that's thoroughly unrealistic of real components.

[2] Positive or negative n: the positive one is all that we're asking for, but the negative one would also be true if we interpret a negative ratio as being that the "dots" are reversed. We didn't even show dots, as the direction of current definition doesn't matter: the resistance absorbs power from the current, depending only on current magnitude.

Q9

a) Power-factor (PF) supplied at the source-terminals (a,b,c).

By Δ-Y conversion, and considering a single-phase equivalent, we get the following seen in one phase:



The power factor seen by the source is then found by the ratio of the resistive part of this impedance to the total impedance. One way to reason this is that for a given current i passing through this series set of components, the total active power is the product of $|i|^2$ and the resistances, and the total reactive power is the product of $|i|^2$ and the inductive reactances (minus any capacitive reactances). The exercises for the ac power topic give practice at this, for series and parallel connections of components.

$$\text{PF} = \frac{P}{|S|} = \frac{R_1 + R_2/3}{\sqrt{(R_1 + R_2/3)^2 + (\omega L_1 + \omega L_2/3)^2}} = \frac{3R_1 + R_2}{\sqrt{(3R_1 + R_2)^2 + \omega^2 (3L_1 + L_2)^2}}$$

Given a balanced load and source, we expect the same ratio of active to apparent power in each phase, so the above is the power factor seen by the source.

b) Find the value C that is needed for each of three Δ-connected capacitors connected at the source, to make the source supply unity power-factor (PF = 1). In other words, this three-phase capacitor should compensate fully for the reactive power supplied to the inductive load and line that were considered in subquestion 'a'.

Using the single-phase equivalent from subquestion 'a', we find that the total reactive power (the imaginary part $\Im\{\}$ of complex power) supplied to the line and load is

$$Q = \Im\left\{3\frac{|U_{\text{phase}}|^2}{Z^*}\right\} = \Im\left\{\frac{3\left(\frac{U}{\sqrt{3}}\right)^2}{(R_1 + R_2/3) - \text{j}\omega(L_1 + L_2/3)}\right\} = \Im\left\{\frac{U^2}{(R_1 + R_2/3) - \text{j}\omega(L_1 + L_2/3)}\right\},$$

$$Q = \frac{U^2\omega(L_1 + L_2/3)}{(R_1 + R_2/3)^2 + \omega^2(L_1 + L_2/3)^2}.$$

The total reactive power consumed by a delta-connected set of capacitors C to line-voltage U is

$$Q_{\text{c}} = \Im\left\{3\frac{U^2}{\left(\frac{1}{\text{j}\omega C}\right)^*}\right\} = -3U^2\omega C,$$

or in other words, the reactive power *generated* by the capacitors is $3U^2\omega C$.

To obtain the required compensation, we need these reactive powers to cancel:

$$Q + Q_{\text{c}} = 0 \quad\implies\quad 3U^2\omega C = \frac{U^2\omega(L_1 + L_2/3)}{(R_1 + R_2/3)^2 + \omega^2(L_1 + L_2/3)^2},$$

which determines the necessary C as

$$C = \frac{L_1 + L_2/3}{3(R_1 + R_2/3)^2 + 3\omega^2(L_1 + L_2/3)^2}.$$

c) For the unbalanced resistive load (in the right-hand diagram), find the magnitude of the largest of the line-currents.

The largest is i_2, with magnitude $|i_2| = \sqrt{7}\frac{U}{R}$.

Between each pair of lines there is a voltage of magnitude U. The current magnitude in line 1 is therefore $|i_1| = \frac{2U}{R}$, and in line 3 is $|i_3| = \frac{U}{R}$. The current magnitude in line 2 is the magnitude of the *phasor sum* of these currents: $i_2 = -i_1 - i_3$, which follows from KCL ($i_1 + i_2 + i_3 = 0$).

From the way the load resistances are connected it is probably clear "by inspection" that the middle line (2) carries the highest current. Perhaps you're not satisfied with that claim: it's sensible to be careful at double-checking. We should take a warning from the fact that if the currents i_1 and i_3 had a phase-difference approaching 180° then their sum would be less than the larger of these values: we'd find that i_1 was the biggest magnitude. It's important to check the phase-angle between these two currents!

Writing the full equations or drawing a phasor diagram is useful. If you're getting quite experienced with three-phase calculations, you'll be familiar with the idea of the three line-voltages (defined e.g. as u_{12}, u_{23} and u_{31}) being phasors that sum to zero and that can be drawn end-to-end as an equilateral triangle in the complex plane; its corners correspond to the potentials of the terminals relative to the mean potential of the three terminals. Each resistor is connected across one of the line voltages, so we could write

$$i_2 = -i_1 - i_3 = -\frac{u_{12}}{R/2} - \frac{u_{32}}{R} = \frac{2u_{21}}{R} + \frac{u_{23}}{R} = \frac{2U\angle\phi}{R} + \frac{U\angle\phi - \pi/3}{R},$$

where ϕ is some arbitrary angle depending on what we choose as the angle reference: setting $\phi = 0$ would be sensible unless we care about the phase of i_2 relative to some special reference. Only the magnitude is needed, so ϕ can be ignored as it shifts all parts of the solution by the same amount.

$$|i_2| = \frac{2U}{R} + \frac{U\angle - \pi/3}{R} = \tfrac{U}{R}\sqrt{\left(2 + 1\cos\tfrac{-\pi}{3}\right)^2 + \left(1\sin\tfrac{-\pi}{3}\right)^2} = \tfrac{U}{R}\sqrt{\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{7}\frac{U}{R}.$$

Here are some further comments about making sense of three-phase circuits where there's nothing said about a neutral point of the source:

This question doesn't tell us that the source has a neutral point: we only know about the *voltages between* the three terminals of the source.

However, to simplify our analysis (to be more like the familiar Y-connected source) we can *define* for ourselves a zero potential between the potentials of the three terminals of the source: i.e. their mean value. In practice, such a reference point could be created by a Y-connected set of identical impedances connected to the source terminals; then a voltage measurement between this point and some other point in the circuit would give the potential of that other point.

We might choose to define the potential of terminal 1 as the zero angle, if that's what we're used to handling: choices of potential and angle reference should fundamentally not affect actual physical statements like the power or the magnitude of current or voltage (difference in potential). Then we'd find that, assuming 1,2,3 phase-rotation, the line-voltage u_{12} is a phasor at 30°, etc.

Alternatively, we could be confident and just start by defining one of the *line*-voltages as the angle reference, without bothering about any concept of potentials. Our circuit has connections only to the three line-conductors between which the voltages are known, so there's no *need* to consider hypothetical reference-points: it just sometimes helps us draw a more familiar diagram.