KTH EI1102 / EI1100 Elkretsanalys Omtenta 2016-03-22 kl 08–13

Tentan har 6 tal i 2 delar: tre tal i del A (15p), tre i del B (15p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ...

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Godkänd tenta kräver minst 25% i del A, 25% i del B, och 50% i genomsnitt (båda delar). Betyget räknas från summan över båda delar, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

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Del A. Likström och Transient

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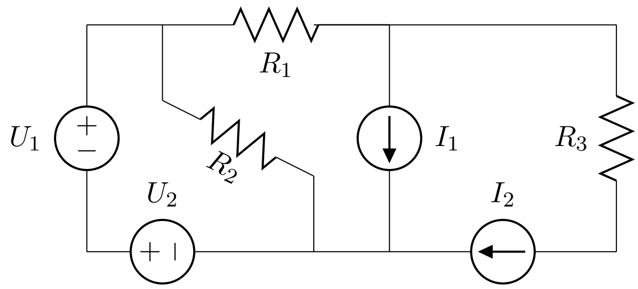
1) [5p] Bestäm effekterna absorberade av de följande komponenterna:

a) [1p] motståndet R_3
b) [1p] motståndet R_2
c) [1p] motståndet R_1
d) [2p] spänningskällan U_2



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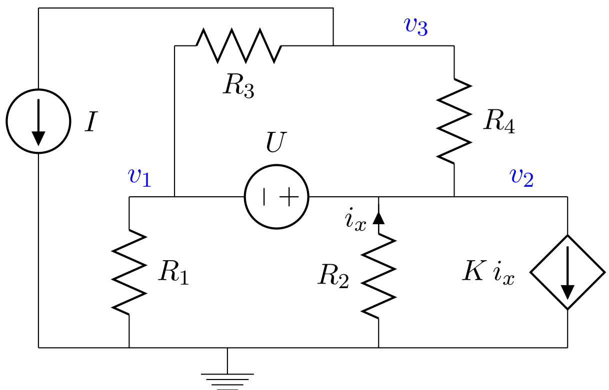
2) [5p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1, v_2, v_3.

Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.



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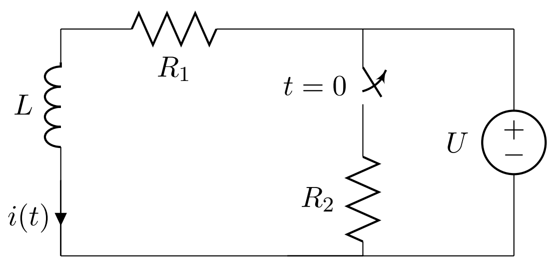
3) [5p]

Bestäm $i(t)$, för $t > 0$.



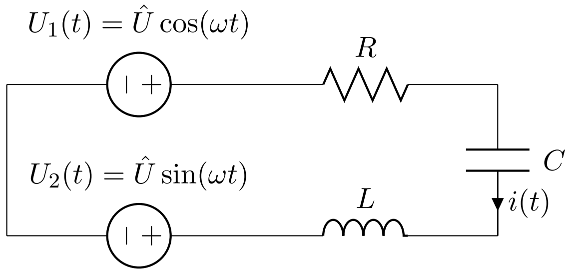
Del B. Växelström

4) [5p]

Bestäm $i(t)$.

Båda källor har samma toppvärde $\hat{U}$.

Ledning: det kanske hjälper att börja med en likströms krets med två källor och tre motstånd, och sedan att byta dessa till fasvektorer och impedanser.



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5) [5p]

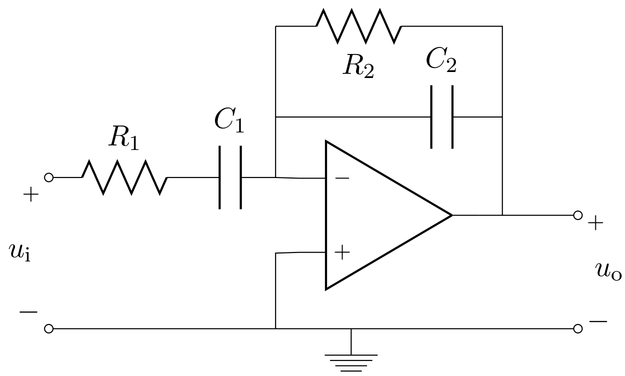
a) [2p] Bestäm kretsens nätverkfunktion,

$$H(\omega) = \frac{u_o(\omega)}{u_i(\omega)}.$$

b) [3p] Svaret till deltal 'a' kan skrivas i den följande formen,

$$H(\omega) = \frac{-j\omega/\omega_1}{(1 + j\omega/\omega_2)(1 + j\omega/\omega_3)}.$$

Skissa ett Bode amplituddiagram av funktionen $H(\omega)$.
Använd funktionen i formen ovan, oavsett vad du fick i deltal 'a'.
Anta att $\omega_1 \ll \omega_2 \ll \omega_3$. Markera viktiga punkter och lutningar.



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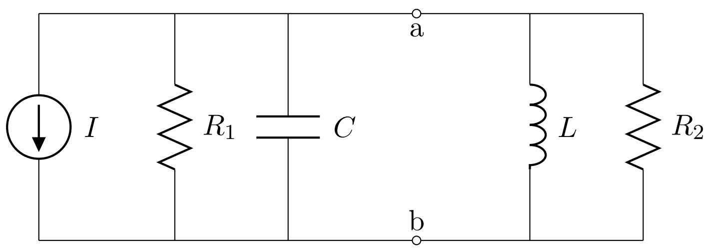
6) [5p]

Strömkällan har vinkelfrekvens ω, och effektivvärde I.

Kretsen till höger om polerna a-b kan betraktas som en last, och kretsen till vänster kan betraktas som en Nortonkälla.

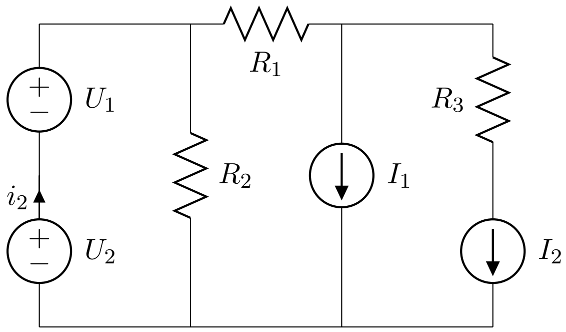


a) [3p] Vilka värde av R_2 och C gör att den största möjliga effekten utvecklas i R_2? Uttryck dessa värde som funktioner av andra komponentvärden.

b) [2p] Vilken effekt levereras till R_2 när komponentvärderna är enligt deltal 'a'.

Solutions (EI1102/EI1100, VT16, 2016-03-22)

Q1



The powers into (absorbed by, or supplied to) the listed components are found by the usual methods:

For a resistor R it is often easiest to find the magnitude of voltage *or* current, then to use the relation $P = i^2R$ or $P = u^2/R$.

For a source, find the quantity that the source *doesn't* determine (a voltage-sources's current or a current-source's voltage) then multiply this with the source value, *taking care about the relative directions* of current and voltage definition.

a) $P_{\mathrm{R3}} = I_2^2 R_3$.
This resistor R_3 is series-connected to current-source I_2, which therefore determines its current.

b) $P_{\mathrm{R2}} = \frac{(U_1+U_2)^2}{R_2}$.
This resistor R_2 is parallel-connected to a voltage that is the sum of U_1 and U_2.
For a more formal justification, consider KVL around the loop of U_1, U_2 and R_2, then Ohm's law in R_2.

c) $P_{\mathrm{R1}} = (I_1 + I_2)^2 R_1$.
KCL above source I_1 gives a current $I_1 + I_2$ flowing left to right through R_1.
The choice of direction doesn't actually matter, as since $(I_1 + I_2)^2 = (-I_2 - I_1)^2$.

d) $P_{\mathrm{U2}} = -U_2 i_2 = -U_2 \left(\frac{U_1+U_2}{R_2} + I_1 + I_2\right)$.
The current in source U_2 (marked as i_2, above) can be found by KCL in three branches: R_2, I_1 and I_2. This is probably most obvious if KCL is done in the bottom node. We have here defined the current *out* of the + terminal of U_2,

$$i_2 = \frac{U_2 + U_1}{R_2} + I_1 + I_2,$$

so if this is multiplied by U_2 it will give the power *supplied* by U_2. We were asked to find the power *absorbed* by source U_2, so a negative sign is included in the final answer,

$$P_{\text{absorbed}} = -U_2 \left(\frac{U_2 + U_1}{R_2} + I_1 + I_2\right)$$

Q2

Extended nodal analysis ("the simple way")

Let's define the unknown current in the voltage sources as i_α, with its positive direction defined into the source's + terminal.

Then KCL (let's take outgoing currents) at all nodes except ground gives:

$$\text{KCL}(1): \quad 0 = \frac{v_1}{R_1} + \frac{v_1 - v_3}{R_3} - i_\alpha \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + \frac{v_2 - v_3}{R_4} + i_\alpha + Ki_x \tag{2}$$

$$\text{KCL}(3): \quad 0 = I + \frac{v_3 - v_1}{R_3} + \frac{v_3 - v_2}{R_4} \tag{3}$$

(4)

The voltage-source introduced the problem of an extra unknown in the KCL equations; it can solve this problem by providing an extra equation without further unknowns:

$$v_2 - v_1 = U \tag{5}$$

The controlling variable of the dependent source needs to be defined in terms of the other known or unknown quantities. Our dependent current-source's controlling variable is a current i_x marked in R_2. This can be described as

$$i_x = -\frac{v_2}{R_2}. \tag{6}$$

$$\tag{7}$$

The above is a sufficient set of equations for a solution.
Lots of other ways of writing the equations are possible, such as eliminating i_x before writing the equations, avoiding i_α by writing just KCL at nodes 1 and 2 together ('supernode'), etc. In many cases the above equations can easily be manipulated into that same form by some subsitutions.

Q3

It's constant: $i(t) = \frac{U}{R_1}$. This can be shown by a methodical way or a quick way! It's nice if you've seen the quick way and had the confidence to claim it: that's shows a good 'conceptual feel'. But there's no deduction for going through a general method, if it is applied correctly.

The quick way is to note that the same voltage exists across the branch of R_1,L at *all* times, so the switch does not affect this current!

"Long ago", before $t = 0$, an equilibrium can be assumed for the inductor's current, where it has reached a steady value. The inductor's voltage is then zero, because this voltage is $L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 0$ is the assumption in equilibrium. At the time of the switch opening, the inductor's current, by KVL around the outer loop, is therefore $i(0) = U/R_1$.

When the switch opens, the voltage source supplies less current, but the branch with the inductor is not affected: it has the same fixed voltage always applied to it.

Hence, $i(t) = \frac{U}{R_1}$ describes the inductor's current for all times after $t = 0$ and in fact it also is true for a long time before that too (if we assume that the circuit was assembled at '$-\infty$').

One longer way is to form an ODE and use the initial condition to solve it.
With the switch open, KVL gives $L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + R_1 i(t) = U$, which has the general solution $i(t) = \frac{U}{R_1} + A\mathrm{e}^{-tR_1/L}$. The initial condition is also found by KVL, as $i(0^+) = i(0^-) = \frac{U}{R_1}$. This allows the constant A to be determined, by requiring our solution to have this value U/R_1 at the initial time: $\frac{U}{R_1} = i(0) = \frac{U}{R_1} + A\mathrm{e}^0$ implies that $A = 0$.

Another way is to find the initial and final values and the time-constant, perhaps by a Thevenin equivalent: after the switch opens the inductor is connected to a circuit that is already a Thevenin source of U and R_1.

The initial and final values are the same, $i(0) = i(\infty) = U/R_1$, as KVL in the outer loop gives the same result regardless of the switch. Then, for a first-order circuit, $i(t) = i(\infty) + (i(0) - i(\infty)) \cdot \mathrm{e}^{-t/\tau} = i(\infty) = U/R_1$. The time-constant is $\tau = L/R_1$, but this is irrelevant as the coefficient for the $\mathrm{e}^{-t/\tau}$ term is zero.

Q4

Both sources have the same magnitude and frequency of sinusoidal voltage; they differ only in phase. Let's take the upper source as the reference angle, i.e. let's use a cosine reference. We'll use the peak values as the phasor magnitudes. The voltage sources are then represented as the following phasors,

$$U_1(\omega) = \hat{U}\angle 0 = \hat{U}, \quad \text{and} \quad U_2(\omega) = \hat{U}\angle -\pi/2 = -\mathrm{j}\hat{U}.$$

The total impedance of the loop is

$$Z = R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C} = R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right).$$

The current phasor is found by "ac Ohm's law",

$$i(\omega) = \frac{U_1(\omega) - U_2(\omega)}{Z} = \frac{\hat{U}\ (1-\mathrm{j})}{R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)}.$$

In polar form this is

$$i(\omega) = \frac{\sqrt{2}\hat{U}\angle -45^\circ}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\angle \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}} = \frac{\sqrt{2}\hat{U}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\angle -\pi/4 - \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

This phasor can now be converted back to a time-function: the same choice of peak value and cosine reference must be used as when we converted from time to phasors,

$$i(t) = \frac{\sqrt{2}\hat{U}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\left(\omega t - \pi/4 - \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}\right).$$

Q5

a) Find the network function $H(\omega) = \frac{u_o(\omega)}{u_i(\omega)}$.

This is a classic configuration of an inverting amplifier.
If we call the input and feedback impedances Z_i and Z_f, then $H = \frac{-Z_f}{Z_i}$.

In our case, $Z_i = R_1 + \frac{1}{\mathrm{j}\omega C_1}$ (series), and $Z_f = \frac{R_2 \frac{1}{\mathrm{j}\omega C_2}}{R_2 + \frac{1}{\mathrm{j}\omega C_2}}$ (parallel).

From the above expressions, or from direct application of KCL at the inverting input, we find

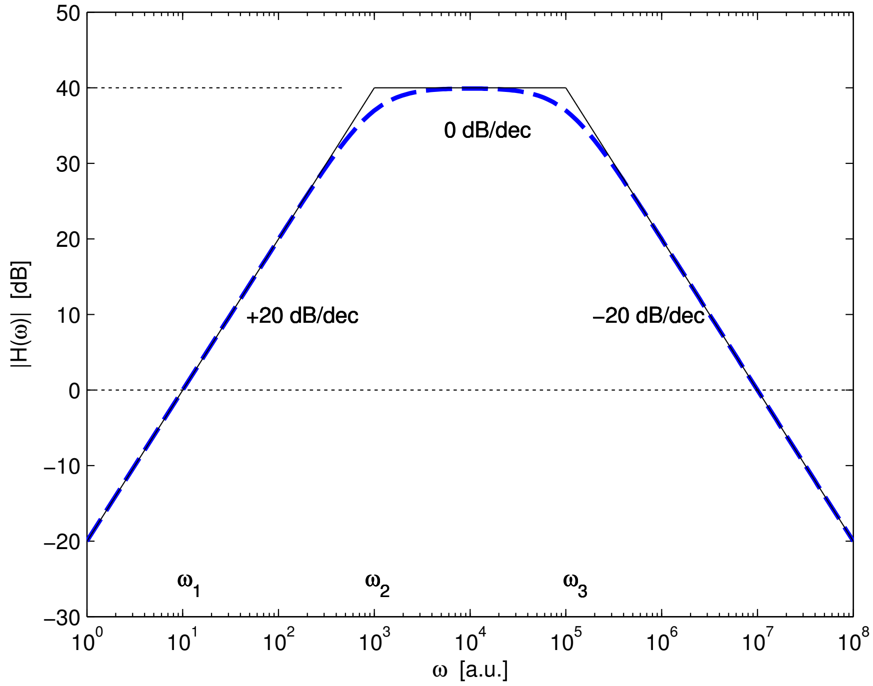
$$H(\omega) = \frac{-\frac{R_2\frac{1}{\mathrm{j}\omega C_2}}{R_2+\frac{1}{\mathrm{j}\omega C_2}}}{R_1 + \frac{1}{\mathrm{j}\omega C_1}} = \frac{-R_2\frac{1}{\mathrm{j}\omega C_2}}{\left(R_1 + \frac{1}{\mathrm{j}\omega C_1}\right)\left(R_2 + \frac{1}{\mathrm{j}\omega C_2}\right)} = \frac{-\mathrm{j}\omega C_1 R_2}{(1+\mathrm{j}\omega C_1 R_1)(1+\mathrm{j}\omega C_2 R_2)}.$$

b) Sketch a Bode amplitude plot of the function

$$H(\omega) = \frac{-\mathrm{j}\omega/\omega_1}{(1+\mathrm{j}\omega/\omega_2)(1+\mathrm{j}\omega/\omega_3)}.$$

Assume $100\,\omega_1 = \omega_2$, and $\omega_2 \ll \omega_3$.

This is shown below: the frequency is in arbitrary units, and the ratio of ω_3/ω_2 has been chosen as 100.



The main features that should be marked are the 40 dB level of the pass-band, the intercept of 0 dB at $\omega = \omega_1$, the the changes of gradient at ω_2 and ω_3, and the gradients of ± 20 dB/decade.

Marking the 0 dB/decade gradient in the pass-band is nice but not necessary, as it's obvious for a flat line! The classic asymptotic Bode amplitude plot has just the straight lines; the further curve shows the actual function plotted numerically.

Note: The negative sign in $H(\omega)$ arises because of this being an inverting amplifier. It has no effect on the amplitude plot, as it simply changes the sign of the function, not the magnitude: $|H(\omega)| \equiv |-H(\omega)|$.

To see how the given $H(\omega)$ can be the same as $H(\omega)$ that was found in subquestion 'a', we can set

$$\omega_1 = \frac{1}{C_1 R_2}, \qquad \omega_{2|3} = \frac{1}{C_1 R_1}, \qquad \omega_{3|2} = \frac{1}{C_2 R_2}.$$

There is nothing to say whether $C_1 R_1$ should correspond to ω_2 or ω_3: the '$_{2|3}$' subscript indicates that either choice could be used, as long as the opposite choice is made for $\omega_{3|2}$.

Q6

This is a classic maximum power situation. It is based on Norton-type sources and parallel components instead of the perhaps more familiar series configuration. We are told that we are free to choose the load resistance and the *source* reactance, instead of the more conventional situation of choosing just the load properties.

a) To the left of the terminals we have a Norton source with current I and admittance

$$Y_1 = \frac{1}{R_1} + \mathrm{j}\omega C.$$

The right is a load, with admittance

$$Y_2 = \frac{1}{R_2} + \frac{1}{\mathrm{j}\omega L} = \frac{1}{R_2} - \mathrm{j}\frac{1}{\omega L}.$$

Maximum power transfer to the load requires that $Y_1 = Y_2^*$. This will result in the reactive components 'cancelling' each other. The result could be seen as parallel resonance, ensuring that no source current is wasted in the reactive components. Or it could be seen as reactive power compensation.

Notice that if we'd worked out impedances instead, the maximum power condition is still $Z_1 = Z_2^*$, but the impedances would be nastier expressions: with parallel connection of, admittance is easier to handle, as the real and imaginary parts contain respectively the R and the C or L value, and not the other.

The maximum power condition requires:

$$Y_1 = Y_2^* \quad \Longrightarrow \quad \frac{1}{R_1} + \mathrm{j}\omega C = \frac{1}{R_2} + \mathrm{j}\frac{1}{\omega L},$$

from which comparison of real parts tells us R_2,

$$R_2 = R_1,$$

and comparison of imaginary parts tells us C,

$$C = \frac{1}{\omega^2 L}.$$

b) With the above choice of components R_2 and C, we expect maximum power transfer, implying a parallel resonance of L and C. These two components can therefore be ignored for calculating the power to the resistors: in resonance a parallel L-C circuit has zero admittance, and therefore no current. Alternatively, you can analyse the complete circuit, and substitute $C = \frac{1}{\omega^2 L}$, in which case the above result should emerge from the algebra.

The current I is then equally split between the source and load resistors, since these have equal value. The current through the load resistor R_2 is therefore $I/2$. By the familiar "power = $|i|^2 R$", the power into the load resistor is

$$P_{\mathrm{R2}} = \left(\frac{|I|}{2}\right)^2 R_2 = \frac{|I|^2 R_2}{4} = \frac{|I|^2 R_1}{4}.$$

The answer can be given in terms of R_1 or R_2, since they are equal, and neither was explicitly stated to be the known or the unknown.

We could assume I to represent just the rms magnitude of the source, in which case the absolute-value symbols ($|I|$) aren't needed; if instead I is a phasor then we need to take its magnitude. Either choice is acceptable since this was not explicitly described in the question.