KTH EI1110 Elkretsanalys (CELTE) TEN1 2016-10-28 kl 08–13

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ... Det behöver *inte* lämnas in.

Tentan har 5 tal i två sektioner: 3 i sektion A (12p), och 2 i sektion B (10p). Godkänd kräver:

$$\frac{\max(a, a_k)}{A} \geq 40\% \qquad \& \qquad \frac{b}{B} \geq 40\% \qquad \& \qquad \frac{\max(a, a_k) + b + p}{A + B} \geq 50\%$$

där A=12 och B=10 är de maximala möjliga poängen från sektionerna A och B, a och b är poängen man fick i dessa respektive sektioner i tentan, a_k är poängen man fick från KS1 vilken motsvarar tentans sektion A, och p är bonuspoäng från hemuppgifterna, motsvarande högst 5% (1,1p); funktionen max() tar den högre av sina argument.

Betyget räknas från summan över båda sektioner, igen med bästa av sektion A och KS1, $\frac{\max(a,a_k)+b+p}{A+B}$. Betygsgränserna är 50% (E), 60% (D), 70% (C), 80% (B), 90% (A).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara **kända** storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller en spänningskälla) antas vara **okända** storheter.

Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner.

Dela tiden mellan talen — senare deltal brukar vara svårare att tjäna poäng på ... fastna inte!

Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Lycka till! Nathaniel Taylor (073 949 8572)

Del A. Likström

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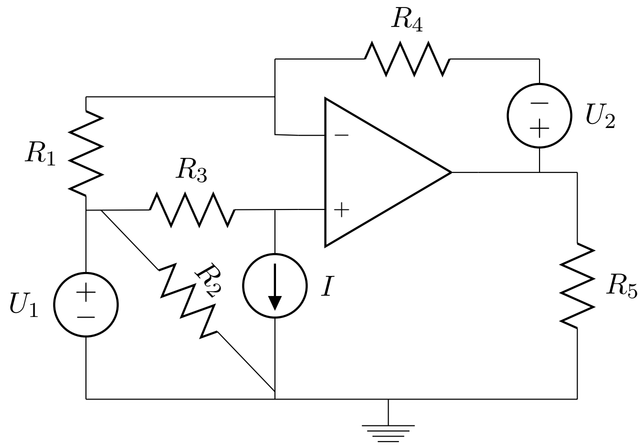
1) [4p] Bestäm följande storheterna:



a) [1p] effekten absorberad av R_2

b) [1p] effekten absorberad av R_3

c) [1p] effekten levererad av U_1

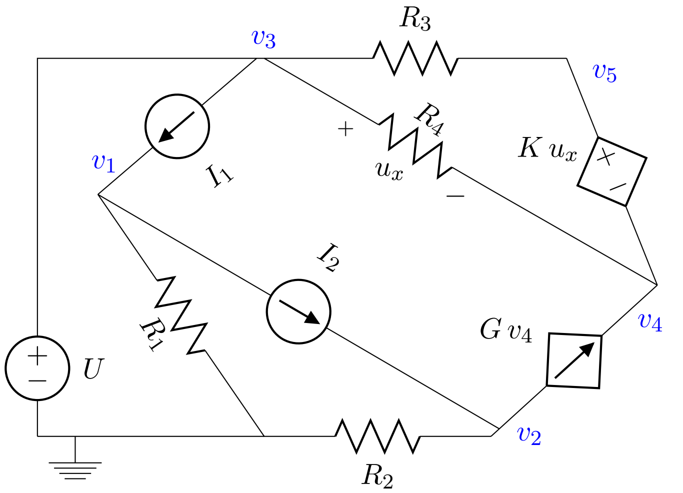
d) [1p] effekten absorberad av U_2

Observera att 'absorberad' eller 'levererad' bara är val av effektens *referens*riktning; vi påstår inte att vi vet på förhand om en källa levererar eller absorberar effekt (den beror ju på komponentvärdena).

2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade potentialerna v_1, v_2, v_3, v_4, v_5.

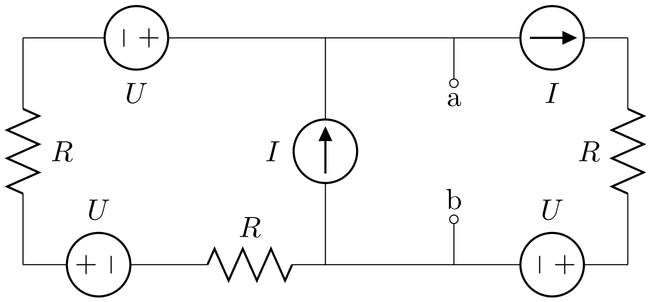
Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.



3) [4p]

a) [3p] Bestäm Theveninekvivalenten av kretsen, med avseende på polerna a-b.

b) [1p] Bestäm den största effekten som kan fås ut från kretsen mellan polerna a-b.



Del B. Transient

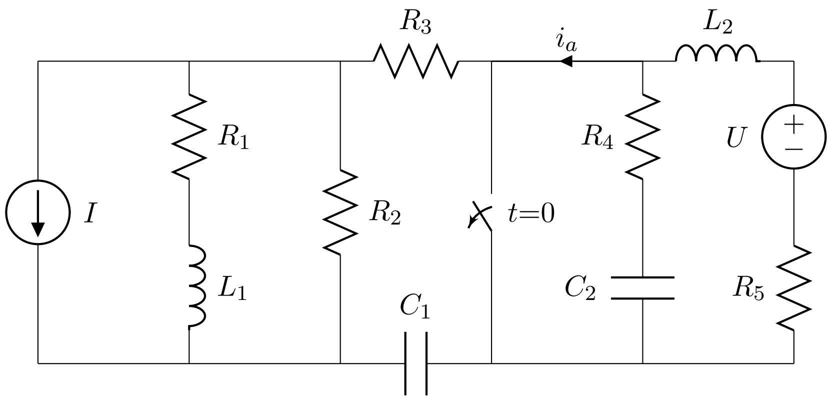
4) [5p]

Bestäm följande storheter, vid de angivna tiderna.

a) [1p] $t = 0^-$
Effekten försörjt av källan I.

b) [2p] $t = 0^+$
Strömmen i_a.

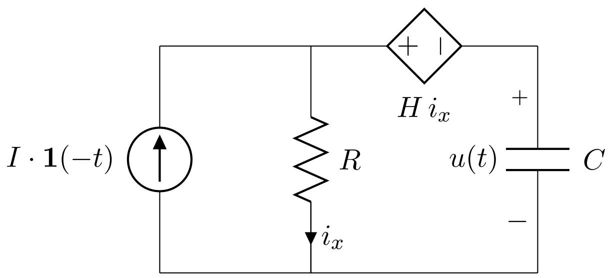
c) [2p] $t \to \infty$
Energin lagrad i L_1 och i C_1.



5) [5p]

Bestäm $i_x(t)$ for $t > 0$.

(Obs. minustecken i stegfunktionens argument!)



Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1110 TEN1 HT16, 2016-10-28)

Q1.

a) $P_{\text{R2,in}} = \frac{U_1^2}{R_2}$
KVL: parallel connection of R_2 and U_1 determines the voltage across R_2.

b) $P_{\text{R3,in}} = I^2 R_3$
KCL at opamp's non-inverting input determines the current through R_3.

c) $P_{\text{U1,out}} = U_1 \left(I + \frac{IR_3}{R_1} + \frac{U_1}{R_2} \right)$ or perhaps you prefer the form $P_{\text{U1,out}} = \frac{U_1^2}{R_2} + U_1 I \left(1 + R_3/R_1\right)$.
The current out of this source's +-terminal, multiplied by the source's voltage, gives the power supplied by the source. By KCL this current is the sum of the currents in R_1, R_2 and R_3. The currents in R_2 and R_3 can be seen based on the previous two solutions. The current in R_1 can be found by KVL, considering that the opamp's inputs have the same potential if it's an ideal opamp with negative feedback: then the voltage across R_1 is IR_3, so the current through R_1 is IR_3/R_1 (left to right).

d) $P_{\text{U2,out}} = \frac{-U_2 I R_3}{R_1}$
The current is R_1 is already found, in part 'c)' as IR_3/R_1. As nothing goes into the opamp input, this must also pass through source U_2, coming out of the source's +-terminal. The energy *absorbed* by the source is then the negation of this current and the source's voltage.

Q2.

Example Method i) **Extended nodal analysis ("the simple way to write")**

Let's define the unknown currents in the voltage sources: i_α into the + terminal of the independent voltage source U, and i_β into the + terminal of the dependent voltage source $K\,u_x$.

Write KCL (let's take outgoing currents) at all nodes except the earth node:

$$\begin{aligned}
\text{KCL}(1): \quad 0 &= \frac{v_1}{R_1} + I_2 - I_1 && (1)\\
\text{KCL}(2): \quad 0 &= \frac{v_2}{R_2} - I_2 + Gv_4 && (2)\\
\text{KCL}(3): \quad 0 &= i_\alpha + I_1 + \frac{v_3 - v_4}{R_4} + \frac{v_3 - v_5}{R_3} && (3)\\
\text{KCL}(4): \quad 0 &= \frac{v_4 - v_3}{R_4} - Gv_4 - i_\beta && (4)\\
\text{KCL}(5): \quad 0 &= \frac{v_5 - v_3}{R_3} + i_\beta && (5)
\end{aligned}$$

Each voltage source relates a pair of node potentials.

$$\begin{aligned}
v_3 &= U && (6)\\
v_5 - v_4 &= K\,u_x && (7)
\end{aligned}$$

The controlling variables of the dependent sources are defined in the circuit diagram by where they are marked, e.g. u_x is the voltage across resistor R_4 with positive reference side towards I_1. We have to define the controlling variables as equations. Although we have two dependent sources, only one of them has a controlling variable specially written for it: the other is controlled by a node potential, which we already have as a variable in our equations. So we just need to defined u_x:

$$u_x = v_3 - v_4 \qquad (8)$$

Example Method ii) **Simplifications (including supernodes) to reduce the equations**

Node 3 can be treated as part of a supernode together with the earth node. Its potential is known to be fixed as U.

Nodes 4 and 5 can be a supernode: we'll define v_4, meaning that node 5 has potential $v_4 + Ku_x$. To avoid this further unknown (u_x) we can express it from the diagram as $v_3 - v_4$, which we can write as $U - v_4$. After all this, the potential at node 5 is:

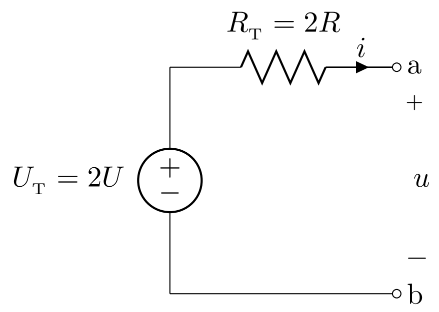
$$\text{KCL}(1): \quad 0 = \frac{v_1}{R_1} + I_2 - I_1 \tag{1}$$
$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} - I_2 + Gv_4 \tag{2}$$
$$\text{KCL}(4\&5): \quad 0 = \frac{v_4 - v_3}{R_4} + (1-K)\frac{v_4 - v_3}{R_3} - Gv_4 \tag{3}$$

To fulfill the requirement (an equation system that could be solved to find all the potentials) we should state the earlier definitions of how the potentials in a supernode relate to each other,

$$v_3 = U \tag{4}$$
$$v_5 = KU + (1 - K)v_4 \tag{5}$$

Q3.

a) The Thevenin equivalent is the following:



If we define as u the voltage of terminal 'a' relative to terminal 'b', and as i the current coming out of the circuit's terminal 'a', then KCL on the three parallel branches within the circuit tells us that

$$\frac{u - 2U}{2R} - I + I + i = 0,$$

from which

$$u = 2U - 2R\,i \qquad \text{(compare to } u = U_T - R_T\, i\text{)}.$$

An alternative way is to 'set independent sources to zero' and find the Thevenin resistance directly, then to find the open-circuit voltage or short-circuit current. We normally suggest finding open-circuit voltage if wanting a Thevenin equivalent, as is the same as the Thevenin voltage so no further algebra is needed. But in this circuit it's also very quick to find the short-circuit current (U/R) and to calculate the Thevenin voltage from this, by $U_T = i_{sc}R_T$.

b) The maximum possible power out from the circuit occurs when the terminal voltage is half of its open-circuit value. Equivalently, we can say it occurs when the current is half of its short-circuit value (this is true precisely when the voltage is half of its open-circuit value).

Hence this maximum power is

$$\frac{u_{\rm oc}}{2}\cdot\frac{i_{\rm sc}}{2}=\frac{U_{\rm T}}{2}\cdot\frac{U_{\rm T}}{2R_{\rm T}}=\frac{U_{\rm T}^2}{4R_{\rm T}}.$$

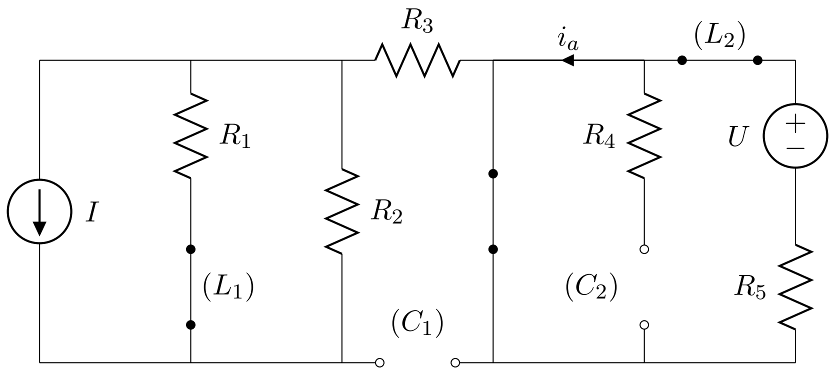
Putting in the values for our circuit,

$$\frac{U_{\rm T}^2}{4R_{\rm T}}=\frac{(2U)^2}{4(2R)}=\frac{U^2}{2R}.$$

Q4.

a) Equilibrium, $t=0^-$.

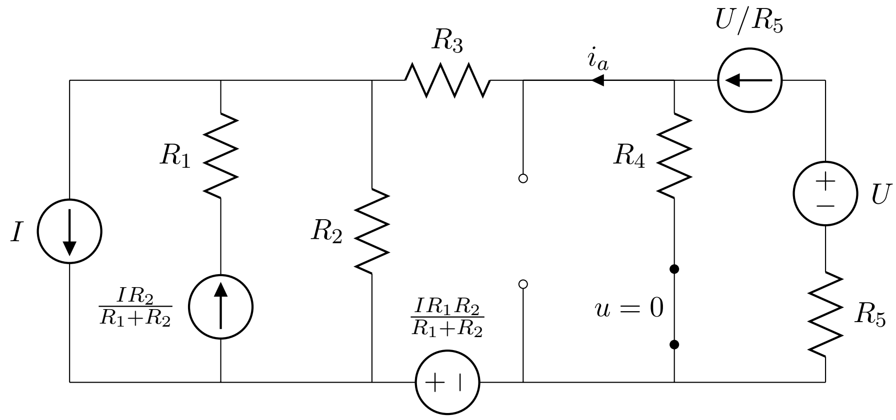
In equilibrium, the subsitutions based on $\frac{\mathrm{d}u(t)}{\mathrm{d}t}=0$ and $\frac{\mathrm{d}i(t)}{\mathrm{d}t}=0$ for all voltages and currents give the following circuit.



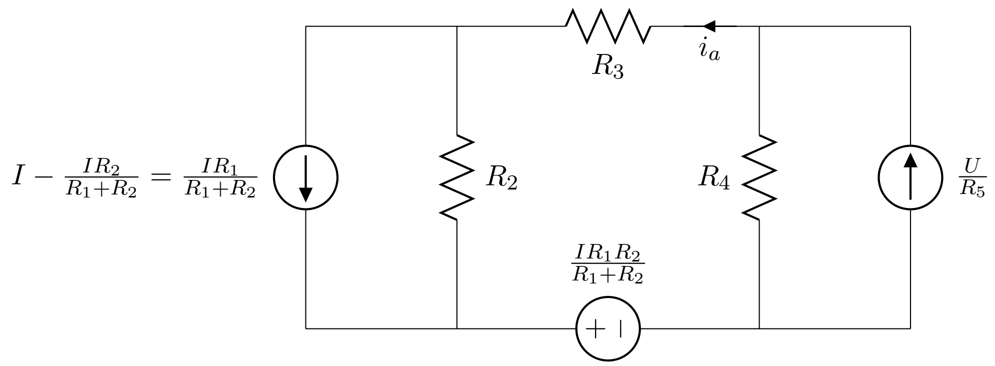
From this, we see the power delivered from source I is just the power it provides to the parallel combination of R_1 and R_2,

$$P_{\rm I}(0^-)=I^2\frac{R_1R_2}{R_1+R_2}$$

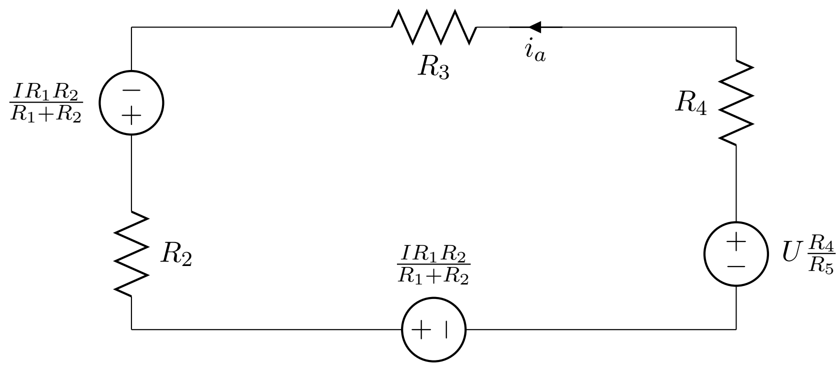
b) Immediately after that equilibrium, by continuity, $t=0^+$.
We can draw the resulting circuit, with the switch being an open circuit, and with capacitors and inductors replaced by their equilibrium values of voltage and current that can be found from the diagram in part 'a)'.



This circuit can be simplified a bit by removing some components that are irrelevant to the sought quantity of i_a, and combining components where possible. The irrelevances in this case are components in series with current sources. The combining is then possible for two parallel current sources.



Now the above circuit tempts us to do some source transformation to get a single loop. (Other options are superposition, or some form of nodal analysis, probably with two unknowns.)



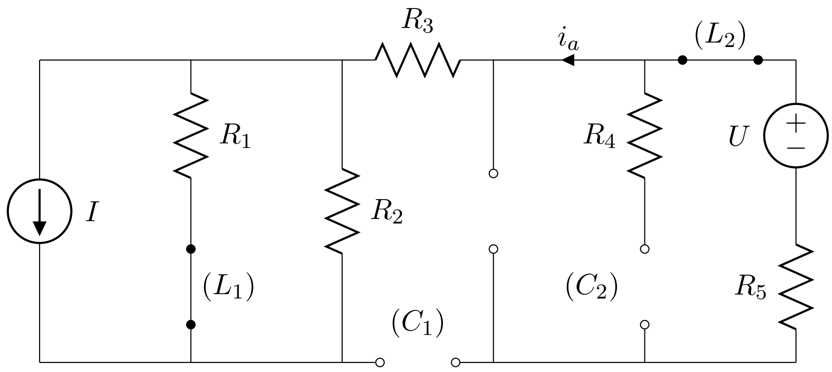
In this simplified circuit, KCL tells us that the same current i_a passes all around the loop.
KVL gives an equation for the current,

$$i_a R_2 - I\frac{R_1R_2}{R_1+R_2} + i_a R_3 + i_a R_4 - U\frac{R_4}{R_5} + I\frac{R_1R_2}{R_1+R_2} = 0 \quad \text{(KVL)}$$

which gives the solution

$$i_a(0^+) = \frac{UR_4}{R_5\,(R_2+R_3+R_4)}.$$

c) Equilibrium, $t \to \infty$.



The energy in L_1 depends on the current in this component: $W_{\text{L1}} = \frac{1}{2}\,L_1\,i_{\text{L1}}^2$.

In this equilibrium state, that current is as in the previous equilibrium (in equilibrium the left part of the circuit is isolated from the right by the open-circuit of C_1; and the left part has not had any change).

$$W_{\text{L1}}(\infty) = \frac{1}{2}L_1\left(\frac{IR_2}{R_1+R_2}\right)^2.$$

The energy in C_1 depends on the voltage across this component: $W_{C1} = \frac{1}{2} C_1 u_{L1}^2$.

In this final equilibrium the voltage if different from the earlier equilibrium: looking at the above diagram we see the switch has a voltage U across it, instead of 0 when it was a short-circuit (closed) at $t < 0$.

Taking KVL around the loop of R_2, R_3, L_2, U, R_5, C_1, we get

$$u_{C1}(\infty) = \frac{IR_1R_2}{R_1+R_2} + 0 + 0 + U + 0 = U + \frac{IR_1R_2}{R_1+R_2}$$

The direction of u_{C1} is not important: it will be squared when we use it to find the energy (only the magnitude of capacitor charging is important). But the relative directions of the KVL components are of course important: we must ensure that the U and $\frac{IR_1R_2}{R_1+R_2}$ terms are taken in the correct relative directions when we go around the loop.

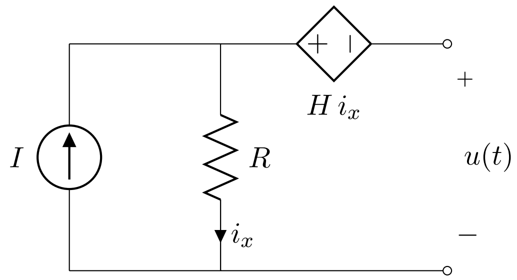
The solution for energy in C_1 is then,

$$W_{C1}(\infty) = \frac{1}{2}C_1\left(U + \frac{IR_1R_2}{R_1+R_2}\right)^2.$$

Q5.

Before the change, $t = 0^-$, there is an equilibrium.

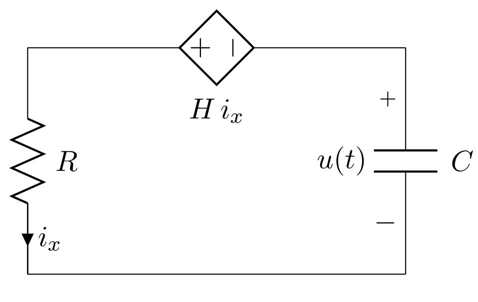
The current source is active, i.e. $I \cdot \mathbf{1}(-t) = I$ at $t = 0^-$.



By KCL the full current I passes down through R, since the capacitor in equilibrium is open-circuit. Hence $i_x = I$, which determines the dependent source's value to be HI.
Then KVL around the right-hand loop gives

$$u(0^-) = I(R - H).$$

After the change ($t > 0$, the period we've been asked about) the current source is zeroed, i.e. $I \cdot \mathbf{1}(-t) = 0$ at $t > 0$. The circuit simplifies to the following, with an initial condition of u known from the above.



By KVL, $Hi_x + u - Ri_x = 0$. This has two unknowns, but as the current i_x is also the current passing upward in the capacitor, we can further include the relation of u and i in a capacitor ($i_x = -C\frac{du}{dt}$) to get a differential equation in just u or just i_x. We'll take the standard approach of solving first for the continuous variable, which is u:

$$-C(H-R)\frac{du}{dt} + u = 0 \qquad \text{i.e.} \qquad \frac{du}{dt} + \frac{1}{C(R-H)}u = 0$$

This has a general solution $u(t) = k\mathrm{e}^{-t/(R-H)C}$.

From the earlier calculation of $u(0^-)$, and by continuity, we can write

$$I(R-H) = u(0^-) = u(0^+) = k\mathrm{e}^{-\frac{0}{C(R-H)}} = k \qquad \text{so} \qquad k = I(R-H).$$

The solution for $u(t)$ is therefore

$$u(t) = I(R-H)\,\mathrm{e}^{-\frac{1}{C(R-H)}t} \qquad (t>0).$$

But it was actually $i_x(t)$ we wanted to find! It is $i_x(t) \;=\; I\,\mathrm{e}^{-\frac{1}{C(R-H)}t} \quad (t>0)$.

One way to find this from $u(t)$ is by KVL,

$$i_xR - i_xH - u = 0 \qquad \Longrightarrow \qquad i_x(t) = \frac{u(t)}{R-H} \;=\; I\,\mathrm{e}^{-\frac{1}{C(R-H)}t} \qquad (t>0)$$

Alternatively we could start from the capacitor equation, being careful about the negative sign that comes from the relative directions in which u and i_x are defined,

$$i_x = -C\frac{\mathrm{d}u}{\mathrm{d}t} = -C\frac{-1}{C(R-H)}\,\mathrm{e}^{-\frac{1}{C(R-H)}t} = I\,\mathrm{e}^{-\frac{1}{C(R-H)}t} \qquad (t>0)$$

* * *

Several variations of this solution could have been used instead.

One is the initial value, final value, time-constant method for first-order linear systems, giving $y(t) = y_\infty + (y_0 - y_\infty)\mathrm{e}^{-t/\tau}$ for some generic quantity y which could represent for example u or i_x.

The time-constant $\tau = C(R-H)$ can be found from the Thevenin resistance R_T of the circuit the capacitor 'sees'. The circuit $(t>0)$ has no independent source, so the open-circuit voltage and short-circuit current are both zero; this prevents us using the classic short-open method of finding R_T. The method of setting independent sources to zero and finding the equivalent resistance is not immediately helpful, since there is a dependent source that will prevent the circuit reducing to just resistors. Writing the relation between i and u at the terminals of the capacitor seems the best way to find R_T. This has basically already been done in the above solution, when we used KVL to give $Hi_x + u - Ri_x = 0$. If we define a current i at the terminals (in the opposite direction to i_x, so that u and i follow the 'active convention' for the Thevenin equivalent) this equation can be written $u = 0 - (R-H)i$, which is the equation of a Thevenin source with $U_\mathrm{T} = 0$ and $R_\mathrm{T} = R - H$.

The initial value of u has already been found at the start. The final value can be found as 0, by considering that there is no indepdenent source, but that there is a path by which the capacitor can discharge.

If one tries to use directly the initial and final values for i_x without using u as an intermediate solution, it is wise to notice that i_x is not the continuous quantity. One should not just assume that $i_x(0^+) = i_x(0^-)$. However, by good luck (!) that equality actually happens to be true in this circuit ... that's why I wrote 'wise' rather than 'important': be careful next time, because another circuit might have a jump of a discontinous quantity!