KTH EI1120 Elkretsanalys (CENMI) KS 2 — 2017-02-17 kl 08–10

Hjälpmedel: Två A4-ark (fyra sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ... De måste *inte* lämnas in.
För den intresserade: två ark för att kunna ha nytt material till KS2 samt lappen som man hade till KS1.

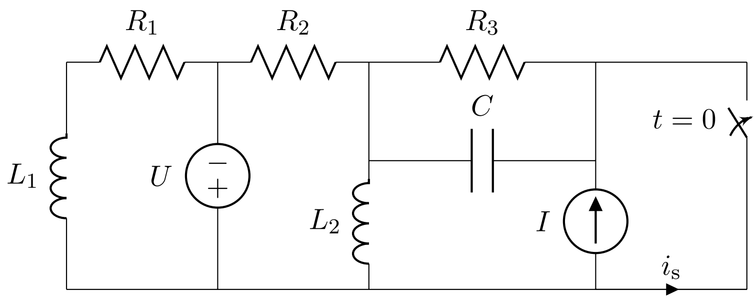
Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas. Var tydlig med diagram och definitioner av variabler.

KS2 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-B i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

Nathaniel Taylor (073 949 8572)

---

1) [5p]
Bestäm följande storheter:



a) [3p] vid $t = 0^+$
(genast efter att brytaren stängs)

$P_{R1}(0^+)$: effekten absorberad av R_1

$P_U(0^+)$: effekten levererad av källan U

$i_s(0^+)$: strömmen i brytaren

b) [2p] vid $t \to \infty$
(lång tid efter att brytaren stängs)

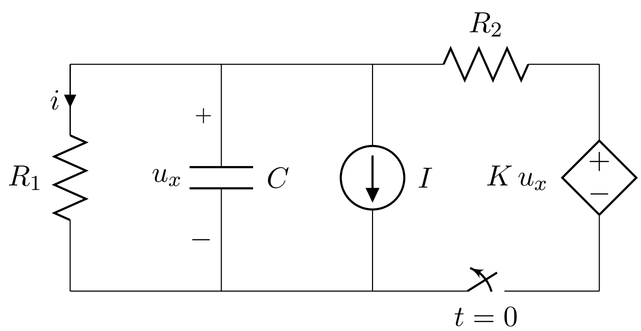
$i_s(\infty)$: strömmen i brytaren

$W_{L2}(\infty)$: energin lagrad i spolen L_2

---

2) [5p]

Bestäm den markerade strömmen i som funktion av tid efter brytaren öppnas:
$i(t)$ för $t > 0$.



Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Short translations of the questions to English:

1. Determine the following quantities:
a) at $t = 0^+$: power absorbed in R_1, power delivered by source U, marked current i_s.
b) at $t \to \infty$: marked current i_s, energy stored in inductor L_2.

2. Find $i(t)$ for $t > 0$, i.e. the marked current i for all time after the switch opens.

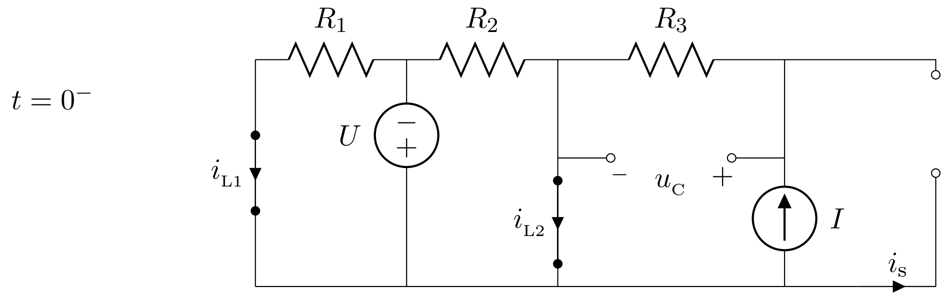
Solutions (EI1120 KS 2 VT17, 2017-02-17)

Q1.

a) $t = 0^+$: Equilibrium and Continuity

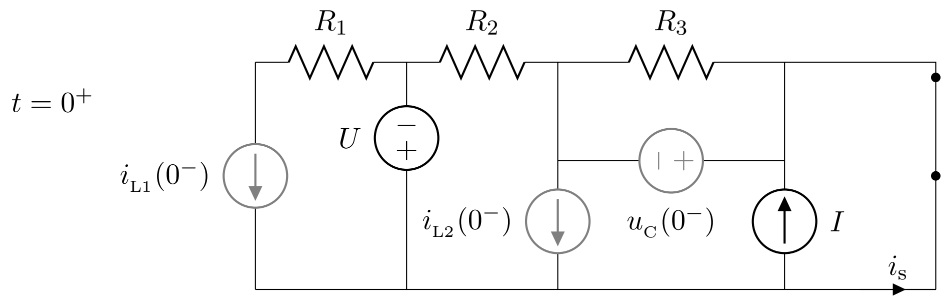
In order to find the requested quantities after the switch closes, $t = 0^+$, we can start with the equilibrium *before* the switch closes, $t = 0^-$, and find all[1] the continuous variables that represent the 'state' (stored energy) of the capacitors and inductors.

For clarity of thinking, we redraw the circuit at this first time-point that we are considering, making everything as simple as possible: the switch is open, and the assumption of equilibrium means that capacitors can be treated as open circuits and inductors as short-circuits. We mark voltages for the capacitors, and currents for the inductors: these are what we will find. Their reference direction can be chosen by us.



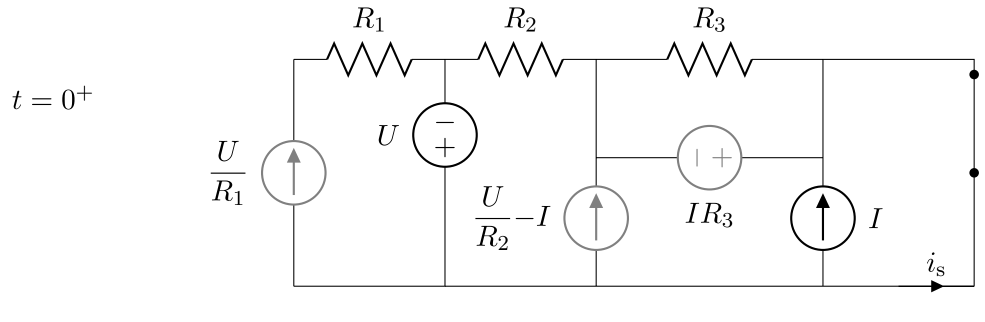
From the above, with our chosen reference directions, we see that:
$i_{L1}(0^-) = -U/R_1$ (KVL in leftmost loop, then simple Ohm and KCL)
$i_{L2}(0^-) = I - U/R_2$ (KVL in loop U, R_2, L_2, and 2 KCL)
$u_C(0^-) = IR_3$ (KCL makes I flow in R_3; then Ohm and KVL on R_3, C)

Now we consider the circuit at the time we're actually interested in: $t = 0^+$. As usual, we re-draw it completely, with everything as simply expressed as possible. The switch is a short-circuit at this time. Capacitors and inductors can be treated as voltage sources and current sources respectively, at just this time, with their values being the voltage or current found at the equilibrium before the change.



[1] We might not *have* to find all of the continuous quantities at 0^-, as it might be that when we look carefully at the circuit at 0^+ we find that some of these quantities are not relevant to what we're trying to find. But here we will not try to be clever in this way: we will find everything.

The expressions we found for the continuous quantities, i_{L1} etc, are quite simple, so we can insert the expressions directly into the diagram We can also avoid some negative signs by changing source directions: this is purely a matter of taste!



This is now a dc circuit in which every component has a known value (in terms of the known component values of the original circuit). It can be solved by our usual dc methods, although it admittedly feels a bit tricky for such a simple-looking circuit!

$$P_{R1}(0^+) \;=\; \left(\frac{U}{R_1}\right)^2 R_1 \;=\; \frac{U^2}{R_1}.$$

This was straightforward, by 'trivial KCL' in the top-left node.

$$P_U(0^+) \;=\; U\left(\frac{U}{R_1} + \frac{U - IR_3}{R_2}\right) \;=\; U^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - UI\frac{R_3}{R_2}.$$

This was not altogether straightforward, unless one is lucky enough to see the right approach soon. The product of this source's voltage U and the current coming out its '+' terminal gives us the power P_U delivered by the source. But what is that current?

Let us consider the node above source U: then the current we're looking for is found by KCL on the currents in R_1 and R_2. The current coming in from R_1 is clear from the previous answer: it is U/R_1. The current in R_2 first might be tempting to try to find by KCL at the right of R_2, but this comes up against the problem of unknown currents in the voltage source IR_3 or in the short-circuited switch. We need to notice that there is a KVL loop around the voltage source, switch (short-circuit), capacitor (modelled as source IR_3) and R_2; from this KVL and Ohm's law we find the current right-to-left in R_2 is $(U - IR_3)/R_2$. Then do KCL above source U, and multiply by U to find P_U.

$$i_s(0^+) \;=\; -I\frac{R_3}{R_2}.$$

One method is KCL in the bottom node,[2] being careful to include the current through the voltage source U, which we found in the previous solution. This gives $\frac{U}{R_1} - \left(\frac{U}{R_1} + \frac{U-IR_3}{R_2}\right) + \frac{U}{R_2} - I + I + i_s = 0$, which simplifies a great deal!

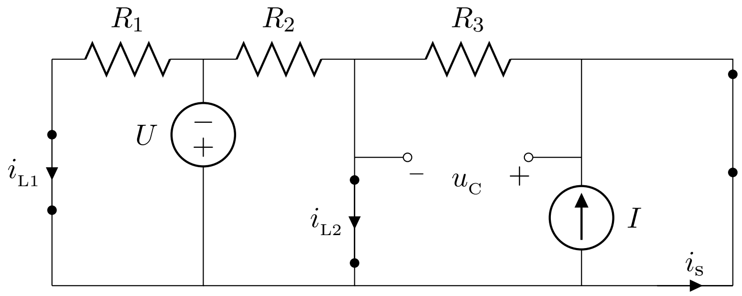
Another method would be KCL above source I. This meets the problem of the unknown current in the capacitor (source IR_3), which can be solved by defining the current and then also taking KCL at the node the other side of R_3. A cleverer way is to see both sides of the source IR_3 as forming a supernode; then the currents in this and R_3 don't even have to be included in the equation: $-i_s - I - \frac{U}{R_2} + I + \frac{U-IR_3}{R_2} = 0$.

[2] You might argue that the closed switch forms a bigger node, consisting of everything from the right of R_3, down through the switch, round to the bottom left of the diagram. That's fine: you can see this all as a big node, or as two nodes joined by a short-circuit (or zero voltage source). We don't have to use KCL only on whole nodes: we can use it on a group of nodes (e.g. when using the supernode method), or on some subpart of a node if the KCL equation includes the currents between that part and other parts of the node. All that matters for KCL is that every current passing a closed boundary in the diagram is taken into account: 'current has to come from somewhere'.

b) $t \to \infty$: Equilibrium

As ever, draw the circuit in the simplest form that we can for this state, the equilibrium a long time after the switch closes. The capacitors and inductors are treated as usual for equilibrium, as we did at $t = 0^-$. The only difference in this circuit at $t \to \infty$ is that switch is closed (short-circuit).

$t \to \infty$



$i_s(\infty) = -I$

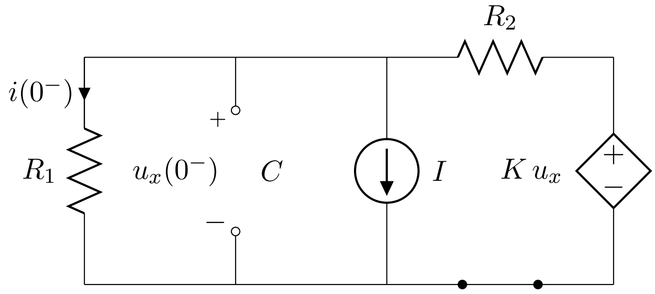
KVL around the loop of L_2, switch, R_3 tells us that there is no voltage across R_3; by Ohm's law there is therefore no current in it, which by KCL above source I tells us that $i_s = -I$.

$$W_{L2}(\infty) = \frac{1}{2} L_2 \left(\frac{U}{R_2}\right)^2$$

KVL around U, R_2, L_2, then Ohm's law, tells us that a current U/R_2 passes right-to-left in R_2. As found in the previous solution, there is no current in R_3. By KCL above L_2, we find $i_{L2} = -U/R_2$. The energy in this inductor is then found from the current.

Q2.

To find the initial condition (the capacitor's voltage at the start of the time we're considering), solve the equilibrium of the circuit before the switch opens $t = 0^-$.



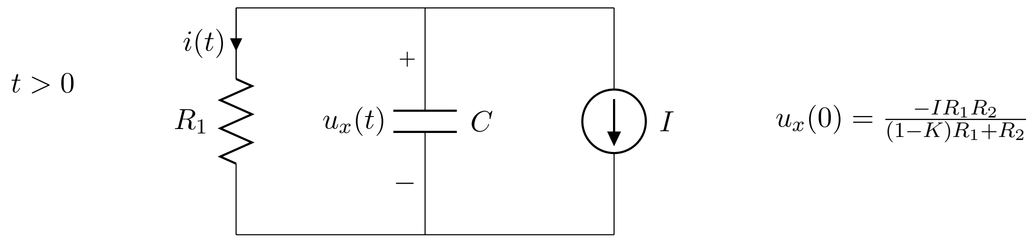
By KCL at the node above the top-left node,

$$\frac{u_x}{R_1} + 0 + I + \frac{u_x - Ku_x}{R_2} \;=\; 0.$$

The capacitor voltage is a continuous quantity, so its value is still the same at the instant after the switch opens. Using this, and rearranging the KCL that was found above for $t = 0^-$,

$$u_x(0^+) = u_x(0^-) = \frac{-I}{\frac{1-K}{R_2} + \frac{1}{R_1}} = \frac{-IR_1R_2}{(1-K)R_1 + R_2}.$$

After the switch opens, the circuit is simplified by the rightmost branch being open-circuited. As this branch cannot have any current, it cannot affect quantities in the other branches in the circuit such as the marked current i that we want to find.



By KCL on the top node,

$$\frac{u_x(t)}{R_1} + C\frac{\mathrm{d}u_x(t)}{\mathrm{d}t} + I = 0 \quad\Longrightarrow\quad \frac{\mathrm{d}u_x(t)}{\mathrm{d}t} + \frac{u_x(t)}{CR_1} = \frac{-I}{C}.$$

This has the general solution

$$u_x(t) = -IR_1 + A\,\mathrm{e}^{-\frac{t}{CR_1}},$$

which can be compared with the initial condition at $t = 0$

$$u_x(0) = \frac{-I}{\frac{1-K}{R_2} + \frac{1}{R_1}} = -IR_1 + A\,\mathrm{e}^0 = -IR_1 + A$$

in order to find the constant A,

$$A = IR_1 - \frac{IR_1R_2}{(1-K)R_1 + R_2} = \frac{IR_1^2(1-K)}{(1-K)R_1 + R_2},$$

from which the solution is

$$u_x(t) \;=\; -IR_1 + \frac{IR_1^2(1-K)}{(1-K)R_1 + R_2}\,\mathrm{e}^{-\frac{t}{CR_1}} \;=\; IR_1\left(\frac{1}{1 + \frac{R_2}{(1-K)R_1}}\,\mathrm{e}^{-\frac{t}{CR_1}} \;-\; 1\right)$$

The quantity we actually want to find is $i(t)$.

A longer method would be to find this from the known $u_x(t)$ would be to use $C\frac{\mathrm{d}u_x(t)}{\mathrm{d}t}$ to find the current in the capacitor, and then KCL to find $i(t)$.

A quicker method – noting that the capacitor and resistor are in parallel, so the voltage $u_x(t)$ is also across the resistor – is direct use of Ohm's law,

$$i(t) = \frac{u_x(t)}{R_1} = \frac{I(1-K)R_1}{(1-K)R_1 + R_2}\,\mathrm{e}^{-\frac{t}{CR_1}} - I \qquad (t > 0).$$

There are several reasonable ways of writing the above expression; some are hinted by the ways that the expressions for u_x were found.

The whole solution could instead have been done by the Thevenin/Norton method and fitting the first-order time-curve from initial value, final value, time-constant.

Or the variable $i(t)$ could have been solved as the dependent variable of the ODE, instead of solving for $u_x(t)$ then finding $i(t)$ from that. One must be careful to remember that if a non-continuous quantity (such as capacitor current) is being solved for, we cannot assume that its value is the same before and after the change, i.e. at times $t = 0^-$ and $t = 0^+$: we would have to find the value at specifically $t = 0^+$ in order to be sure we have the right initial value of what the quantity is doing after the change in the circuit. In our particular case Ohm's law links our quantity $i(t)$ in direct proportion to the continuous quantity $u_x(t)$, so there is no such issue. If instead we had the task of finding the current in the capacitor, the above-mentioned trap might catch us if we're not careful: the capacitor's current is 0 at $t = 0^-$, but jumps to a different value $(-IR_1(1-K)/(R_1(1-K) + R_2))$ immediately after, i.e. $t = 0^+$.