KTH EI1120 Elkretsanalys (CENMI) TEN1 2017-03-16 kl 08–13

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Upp till tre A4-ark (sex sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek. De måste *inte* lämnas in.

För den intresserade: tre ark för att man kan välja att återanvända vad man hade till KS1 och KS2, samt att lägga till en ny för växelström.

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. *Senare deltal brukar vara svårare* att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_k vara poängen man fick från kontrollskrivning KS1, och b_k poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a, a_\mathrm{k})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b, b_\mathrm{k})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a, a_\mathrm{k}) + \max(b, b_\mathrm{k}) + c + h}{A + B + C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

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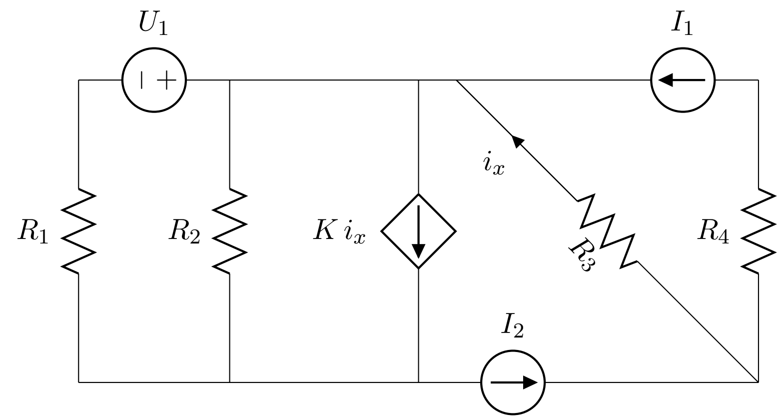
Del A. Likström

1) [4p] Bestäm effekterna som försörjs från kretsen till följande motstånd:

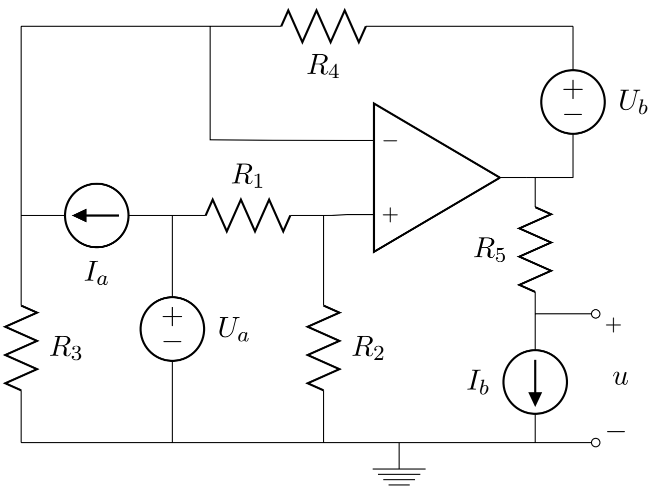
a) [1p] R_4.

b) [1p] R_3.

c) [2p] R_2.



2) [4p]

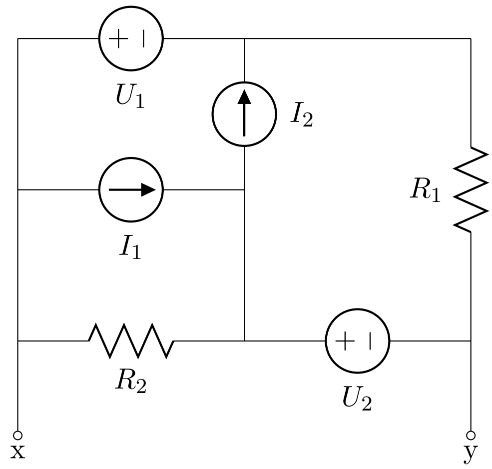


Bestäm u genom valfri metod; alternativt, använd nodanalys för att skriva ekvationer som skulle kunna lösas för att bestämma u.

Hela poäng erhålls vid korrekt lösning av u. Vid korrekta ekvationer utan hela lösningen av u erhålls 0,5p mindre.

Observera att du kan börja med att använda nodanalys för att få ekvationerna, och sedan försöka lösa dessa för full poäng. Då får du en trygghet i att ekvationerna ger flest poäng även om du inte lyckas lösa dem.

3) [4p]



a) [3p] Ett motstånd R_x ska kopplas mellan polerna x-y i kretsen till höger. Vilket värde ska det ha för att få ut den maximala möjliga effekten från kretsen?

b) [1p] Till skillnad från deltal 'a' ska en spänningskälla U_x, i stället för motståndet R_x, anslutas mellan x-y (med källans '+' pol till pol x). Vilket värde ska U_x ha för att få ut maximaleffekt från kretsen?

Short translations of Section-A questions to English:

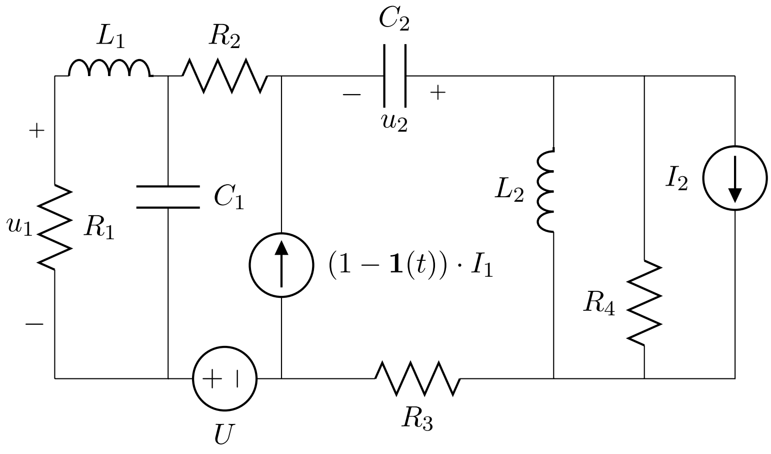
1. Find the power delivered to each of: a) R_4, b) R_3, c) R_2.

2. Two *alternative* options: [for all 4 points] solve for u completely, by any method you wish; or [for 3.5 points] use nodal analysis to write (but not have to solve) a set of equations that could be solved to find u.

3. A maximum power and two-terminal equivalent question.
a) A resistor R_x is connected between x-y: what value should R_x be in order to obtain the maximum possible power from the shown circuit?
b) A voltage source U_x is connected at x-y (instead of the resistor R_x from part 'a'); its direction is with '+' terminal connected to terminal x. What value should U_x be in order to obtain maximum power from the shown circuit?

Del B. Transient

4) [5p] Bestäm följande:



a) [1p] Vid $t = 0^-$,
effekten levererad av strömkällan I_1.

b) [3p] Vid $t = 0^+$,
spänningen u_1 över R_1,
energin lagrad i kondensatorn C_2,
effekten absorberad i motståndet R_2.

c) [1p] Vid $t \to \infty$,
spänningen u_2 över kondensatorn C_2.

Obs: $\mathbf{1}(t)$ är enhetsstegfunktionen.

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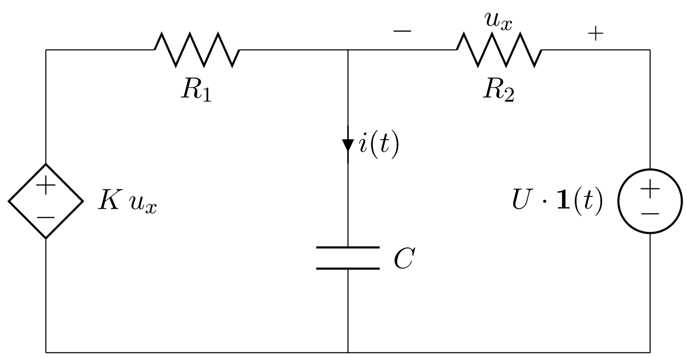
5) [5p]

Bestäm $i(t)$, för $t > 0$.



Short translations of Section-B questions to English:

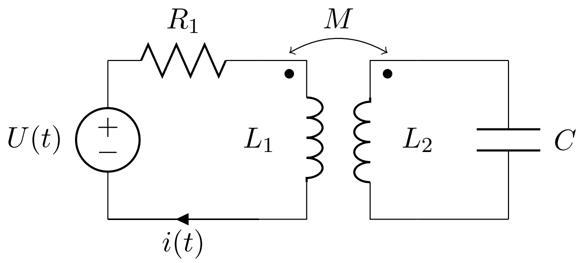
4. Find the following quantities:
a) at $t = 0^-$: power delivered by source I_1.
b) at $t = 0^+$: voltage u_1, energy stored in C_2, power absorbed by R_2.
c) as $t \to \infty$: voltage u_2.

5. Determine $i(t)$, for $t > 0$.

Del C. Växelström

6) [4p]

Kretsen innehåller två kopplade spolar, med ömsesidiginduktans M. Spänningskällan beskrivs med tidsfunktionen $U(t) = \hat{U}\sin(\omega t)$.



a) [2p] Bestäm $i(t)$ med villkoret $M = 0$.
Tips: i så fall är spolarna inte kopplade: en mycket förenklad krets kan analyseras.

b) [2p] Bestäm $i(t)$ *utan* villkoret ovan (lösningen kommer nu att bero även på M, L_2 och C).

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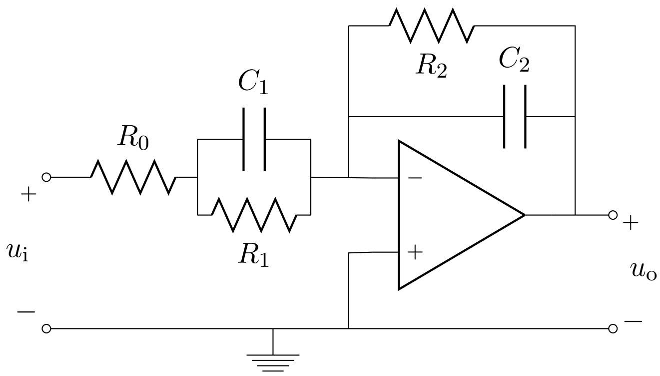
7) [5p]



a) [2p] Bestäm kretsens nätverksfunktion,

$$H(\omega) = \frac{u_\mathrm{o}(\omega)}{u_\mathrm{i}(\omega)}.$$

b) [1p] Visa att svaret till deltal 'a' kan skrivas i den följande formen,

$$H(\omega) = \frac{-K\,(1 + \mathrm{j}\omega/\omega_a)}{(1 + \mathrm{j}\omega/\omega_b)(1 + \mathrm{j}\omega/\omega_c)}.$$

c) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'.
Anta att: $K = 0{,}1$, $100\,\omega_a = \omega_b$, och $\omega_b < \omega_c$.
Markera viktiga frekvenser (t.ex. ω_1), nivåer (t.ex. x dB) och lutningar (t.ex. y dB/dekad).

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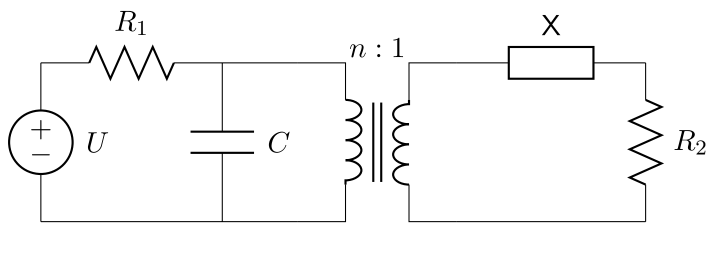
8) [3p]



Källan är en växelspänningskälla med vinkelfrekvens ω.

Motståndet R_2 och komponenten 'X' kan väljas; andra komponenter i kretsen har fasta värden.

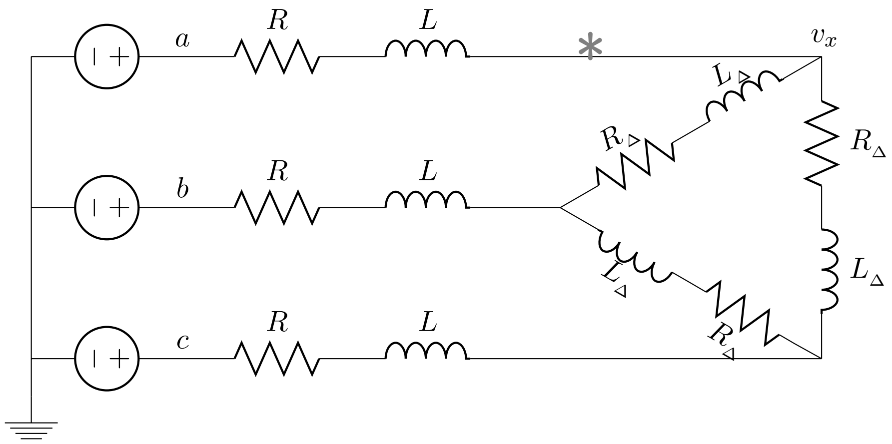
För att få den maximala möjliga effekten (aktiv effekt) levererad till R_2:

a) [1p] vilken komponenttyp (C, L, eller R) ska komponenten 'X' vara?

b) [1p] vilket värde ska den ha?

c) [1p] och vilket värde ska R_2 ha?

9) [6p]



Spänningskällorna utgör en balanserad trefaskälla med huvudspänning U och vinkelfrekvens ω.

Komponenterna R_Δ och L_Δ modellerar faserna av en last, och komponenterna R och L modellerar fasledarna mellan källan och lasten.

a) [2p] Bestäm lastens skenbara effekt $|S|$ och effektfaktor PF, med antagandet $R = 0$ och $L = 0$.

Observera! Antagandet från deltal 'a' ska *inte* användas härefter!

b) [2p] Bestäm potentialen v_x, med antagandet att potentialen i nod 'a' har vinkeln noll, $\angle v_a = 0$.

c) [2p] Ledaren i fas-a blir bruten (öppen krets) vid stjärntecknet '*': vad är $|v_x|$ nu?

Short translations of Section-C questions to English:

6. The circuit has two inductors, between which there is mutual inductance M.
The voltage-source is described by $U(t) = \hat{U}\sin(\omega t)$.
a) assuming $M = 0$, determine $i(t)$; notice that this assumption removes the effect of coupling between the coils, making the solution much simpler.
b) determine $i(t)$ *without* that assumption; the solution will now also depend on M, L_2 and C.

7. An opamp-based filter.
a) determine the function $H(\omega) = u_o(\omega) / u_i(\omega)$.
b) show that the above function $H(\omega)$ can be written in the given form (see Swedish text).
c) Sketch a Bode amplitude-plot of the function from part 'b'. Assume $K = 0.1$, $100\,\omega_a = \omega_b$, $\omega_b < \omega_c$.
Indicate known values of frequencies (e.g. ω_1), levels (e.g. x dB) and gradients (e.g. y dB/decade).

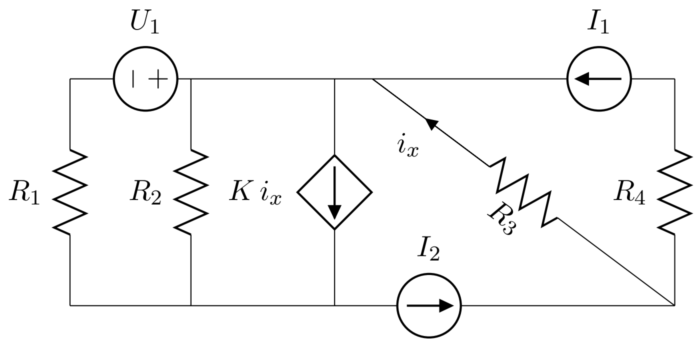
8. The source is an ac voltage-source with angular frequency ω.
Resistance R_2 and the component 'X' can be chosen. Other component values are fixed.
The task is to obtain the maximum possible power (active power) into R_2. To achieve this:
a) What component type (C, L or R) should component 'X' be?
b) What value should it have?
c) What value should R_2 have?

9. The sources form a balanced three-phase source with line-voltage magnitude U and angular frequency ω.
Each $R\,L$ branch represents one phase of a power line; each $R_\Delta\,L_\Delta$ branch represents one phase of a load.
a) Find the apparent power $|S|$ into the load, and the load's power factor, assuming $R = 0$ and $L = 0$.
b) Find potential v_x, assuming that the potential at node 'a' has zero angle, i.e. $\angle v_a = 0$.
c) The conductor of phase-a in the line becomes broken (open) at the asterisk '*': determine $|v_x|$.

Slut … *men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren!*

Solutions (EI1120 TEN1 VT17, 2017-03-16)

Q1



a) $P_{\text{R4}} = I_1^2 R_4$
This resistor is series-connected to the current source I_1, so by KCL its current is also I_1.

b) $P_{\text{R3}} = (I_1 - I_2)^2 R_3$
KCL at the bottom-right node determines the current in R_3 as the difference between I_1 and I_2. The direction ($I_1 - I_2$, or $I_2 - I_1$) does not matter, as the current is squared in order to find the power.

c) $$P_{\text{R2}} = \frac{1}{R_2} \cdot \left(\frac{\frac{U_1}{R_1} + I_2 + K(I_1 - I_2)}{\frac{1}{R_1} + \frac{1}{R_2}} \right)^2 = R_2 \left(\frac{U_1 + KI_1R_1 + (1-K)I_2R_1}{R_1 + R_2} \right)^2$$

This last question is more difficult.
KCL (nodal analysis) or source-transformation appear good methods.

Method 1: KCL, no simplification.

Define voltage u across R_2, with + upwards. At the node above R_2, KCL can be written as

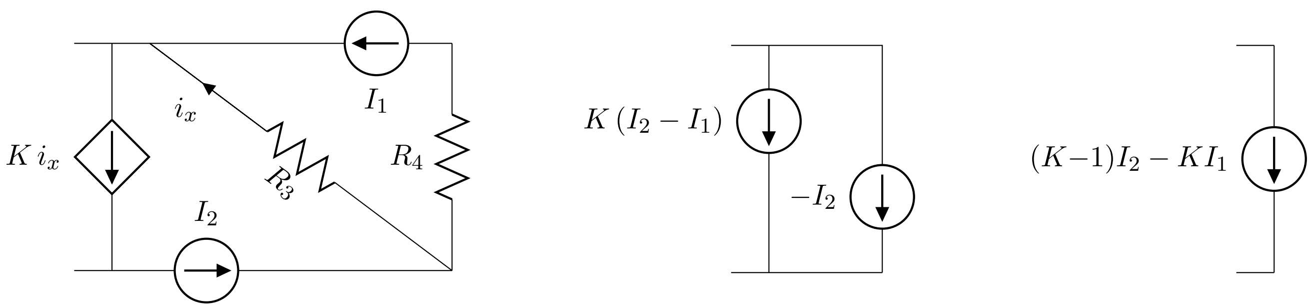
$$\frac{u - U_1}{R_1} + \frac{u}{R_2} + Ki_x - i_x - I_1 = 0 \quad\Longrightarrow\quad u\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{U_1}{R_1} + I_1 - (K-1)(I_2 - I_1)$$

from which

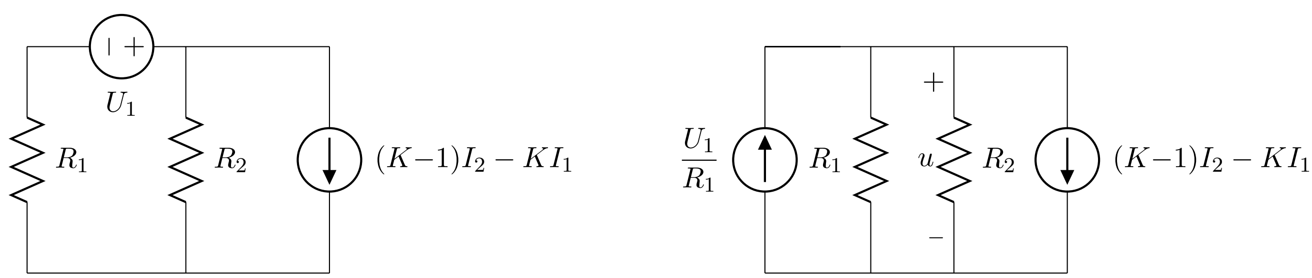
$$u = \frac{R_2}{R_1 + R_2}\Big(U_1 + (1-K)R_1I_2 + KR_1I_1\Big).$$

The power in R_2 can be found as $P_{\text{R2}} = u^2/R_2$, which simplifies to the solution for P_{R2} given above.

Method 2: Transformations.
Some simplification of the right-hand half of the circuit may be helpful:



Then, by a source-transformation on the Thevenin-type source at the left, the voltage u across R_2 can be found from the total current passing through the parallel combination of the two resistances:



Here,

$$u = \left(\frac{U_1}{R_1} - \left((K-1)I_2 - KI_1\right)\right)\frac{R_1R_2}{R_1+R_2} = \frac{R_2}{R_1+R_2}\left(U_1 + (1-K)R_1I_2 + KR_1I_1\right)$$

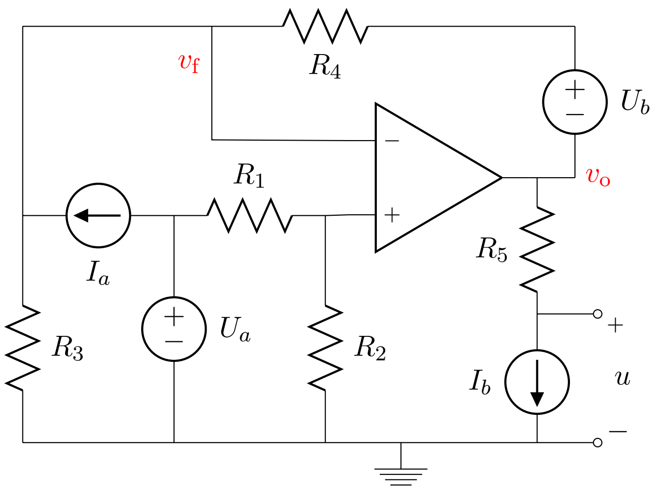
which is – as it should be – the same as by the direct KCL method shown earlier.

Q2

This question was allowed to be treated like the traditional question of “write equations based on nodal analysis, that could be solved, but you don’t need to solve them”, in which case it would be limited to 0.5p less than the full 4p.

For the full points, a complete solution was needed for the single marked quantity u. This solution could be obtained from an equation system, or could be found by any other method. Starting from nodal-analysis based equations seems a wise choice, in order to get some points even if not managing to solve all the way. We show several possible methods here.

Method 1.



Let’s start by a direct method for getting a full solution of u. We’ll describe it in a way loosely based on nodal analysis, using supernodes and other simplifications to reduce the number of equations we need to handle.

If we find the opamp’s output potential, v_o, then the marked voltage u is easily found by KCL and KVL in the branch of R_5 and I_b, as

$$u = v_o - I_bR_5.$$

At the non-inverting input, voltage division gives the potential as $\frac{R_2}{R_1+R_2}U_a$. Voltage division is valid because we know that no current flows in the opamp’s input: so R_1 and R_2 can be treated as series connected. The potential we’ve found is independent of other parts of the circuit, as the voltage source U_a fixes the potential across R_1 and R_2 regardless of I_a.

The negative feedback around the ideal opamp ensures that the output voltage will be whatever is needed to hold the inverting input to the *same potential* as the non-inverting input,

$$v_f = \frac{R_2}{R_1+R_2}U_a.$$

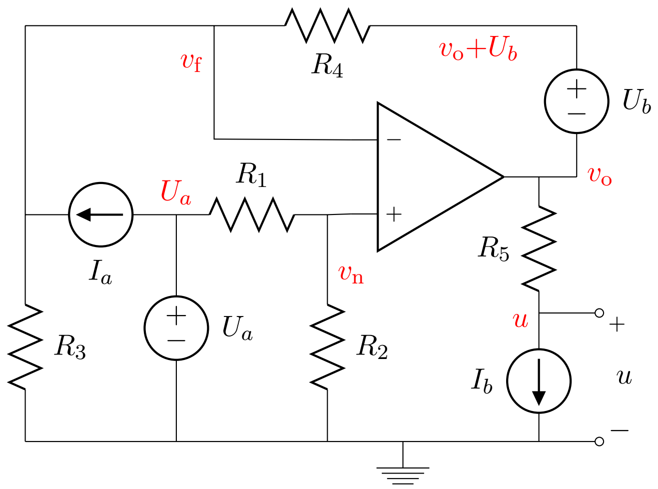
By KCL in the node marked v_f,

$$\frac{v_f}{R_3} - I_a + \frac{v_f - v_o - U_b}{R_4} = 0 \qquad \Longrightarrow \qquad v_o = \left(1 + \frac{R_4}{R_3}\right)v_f - I_aR_4 - U_b.$$

By taking the above three equations for u, v_f and v_o, and eliminating v_f and v_o, the solution is

$$u = \frac{R_3+R_4}{R_3}\cdot\frac{R_2}{R_1+R_2}U_a - U_b - I_aR_4 - I_bR_5.$$

Method 2.



Now we will try nodal analysis, based on supernodes but without any other simplifications such as the voltage-division step used in the above method.

We assign potentials to all the nodes, using already-defined quantities where possible: the potential above the source I_b already has a name, u.

Within a supernode, potentials are expressed relative to one of the nodes in that supernode, so that new variables aren't needed.

Here, the node above source U_a is part of the earth supernode, so it can be defined as $0+U_a$.

The node above the opamp output is $v_\mathrm{o}+U_b$. This is also part of the earth supernode, as we see the opamp output as being a voltage source whose other terminal connects to the earth node. (However, we do not directly know this source's voltage, so we get an unknown potential v_o. The extra knowledge that the opamp's inputs have the same potential is what gives us enough equations to be able to solve for the potentials.)

The nodes at the opamp inputs can be marked with unknown potentials v_f and v_n. We know these are the same potential, but we can start out with different names to make it clearer where the equations came from.

Now we write KCL for any [super]node that is not the earth [super]node. In our circuit, this is just three nodes, since both supernodes are connected with the earth node.

$$\mathrm{KCL}(v_\mathrm{n})_{(\mathrm{out})}: \quad 0 \quad = \quad \frac{v_\mathrm{n}-U_a}{R_1}+\frac{v_\mathrm{n}}{R_2} \tag{1}$$

$$\mathrm{KCL}(v_\mathrm{f})_{(\mathrm{out})}: \quad 0 \quad = \quad \frac{v_\mathrm{f}}{R_3}+\frac{v_\mathrm{f}-v_\mathrm{o}-U_b}{R_4}-I_a \tag{2}$$

$$\mathrm{KCL}(u)_{(\mathrm{out})}: \quad 0 \quad = \quad I_b+\frac{u-v_\mathrm{o}}{R_5}. \tag{3}$$

In this system we have 4 unknowns and 3 equations, but we know also that

$$v_\mathrm{f} \quad = \quad v_\mathrm{n}. \tag{4}$$

We can include this as a fourth equation, or just substitute it into equation (1) to eliminate v_n.

Then, for this particular circuit, the equations can be solved conveniently, eliminating one variable at a time, without needing a simultaneous solution. In fact, this process is the same algebra as what we did in 'Method 1' when we started with equations from voltage division, KCL and KVL.

Method 3.

It would also be acceptable use the 'extended nodal analysis' method without any supernode simplifications. One would define every node potential and a current in every voltage source (including the opamp output), and then write the 6 KCL equations for all nodes except the earth node, along with 3 further equations for the voltage sources. This is in the style of answers to 'Q2' of this course's exam in several previous years. The method involves more writing but less careful thinking. The equations it produces would require more work for solving to find u.

Q3

a) $R_x = \dfrac{R_1 R_2}{R_1 + R_2}.$

The maximum power is obtained when the connected resistor has the same resistance as the source-resistance (Thevenin or Norton resistance) of the circuit that the power is coming from.

We were only asked what resistance should be chosen in order to obtain the maximum power; we do not need to find how big this maximum power is. So we only need to find the source resistance of the circuit, not its Thevenin voltage or Norton current.

For this circuit with no dependent sources, it's easiest simply to "set all the [independent] sources to zero" and simplify the remaining resistors to find the equivalent resistance. This results in R_1 and R_2 in parallel, between terminals x-y, leading to the above expression for R_x.

b) $U_x = \dfrac{U_1 R_2 + U_2 R_1 + (I_2 - I_1) R_1 R_2}{2\,(R_1 + R_2)}.$

In the situation where maximum power is transferred from a linear two-terminal circuit, the terminal voltage is half of its open-circuit value.

The open-circuit voltage between x-y can be found several ways; here we use nodal analysis with a single non-earth supernode (just one KCL).

Define node 'y' as the earth node. We get two supernodes, that include all the nodes in the circuit. At the supernode that includes node 'x', KCL gives

$$\frac{v_x - U_2}{R_2} + I_1 - I_2 - \frac{v_x - U_1}{R_1} = 0,$$

from which

$$v_x \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{U_1}{R_1} + \frac{U_2}{R_2} + I_2 - I_1,$$

giving

$$v_x = \frac{U_1 R_2 + U_2 R_1 + (I_2 - I_1) R_1 R_2}{R_1 + R_2}.$$

We defined node 'y' as the reference, so this potential v_x is also the open-circuit voltage between x-y. Maximum power output of the circuit implies that the 'thing' that we connect to take power from terminals x-y should bring the voltage down to half of the open-circuit value. As we are considering the 'thing' being a voltage source, we need it to have a voltage of $v_x/2$ in the same direction as v_x relative to the marked terminals.

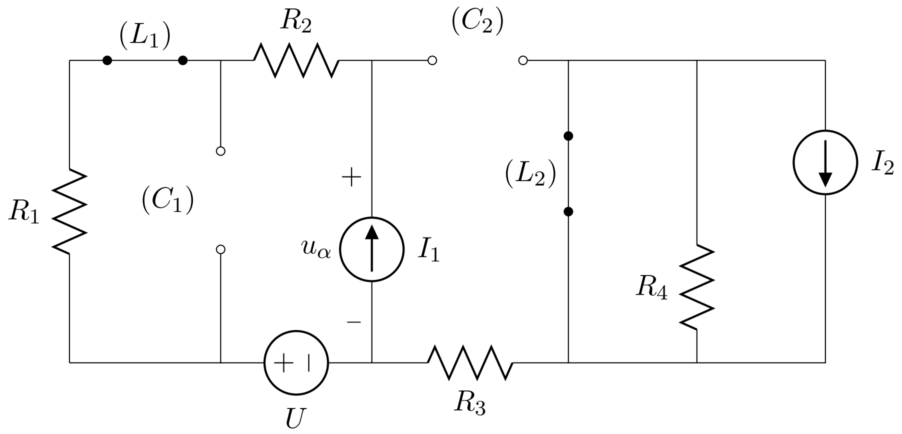
Q4

a) This time is just before the step function changes: the left current source still has a value of I_1.

The circuit is in equilibrium, as it has been unchanged over all previous time (and the circuit is assumed to be stable).

From the assumption of equilibrium, we expect zero voltage across inductors, and zero current in capacitors. The currents and voltages in the circuit at this time can therefore be found by analysing the simplified diagram shown below.

The unknown voltages across capacitors and currents in inductors have also been marked in the diagram: these are not needed for the Q4a solution, but they are useful for Q4b.



By finding the voltage u_α across the current source I_1, we can find the power delivered by that source. KVL around the leftmost complete loop (I_1, R_2, R_1, U) determines this voltage:

$$u_\alpha = U + I_1(R_1 + R_2).$$

The relative directions of I_1 and u_α follow the 'active convention', so the power delivered by source I_1 is

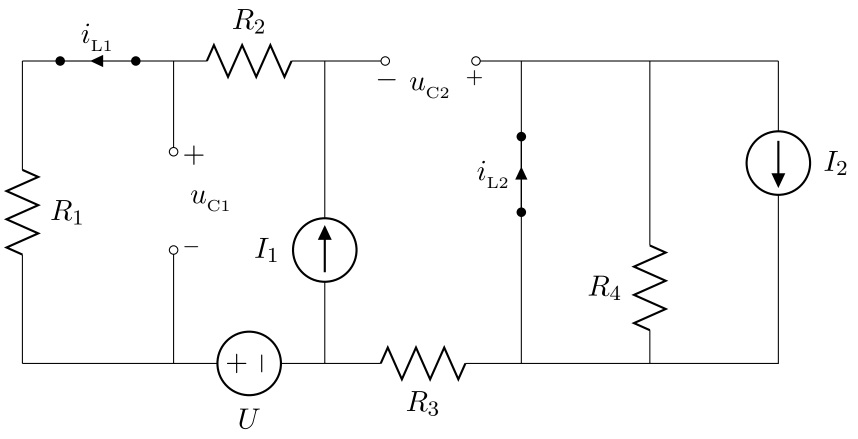
$$P_{\mathrm{I1}}(0^-) \;=\; u_\alpha I_1 = I_1\big(U + I_1(R_1 + R_2)\big) \;=\; UI_1 + I_1^2(R_1 + R_2)$$

b) At this time, $t = 0^+$, a change has newly occurred: the current from the left current source has changed from I_1 to 0, because of the step function.

In order to solve for this new state we cannot still assume the capacitors and inductors to be open- and short-circuits: their currents and voltages (respectively) could have changed instantaneously when the current from the source changed.

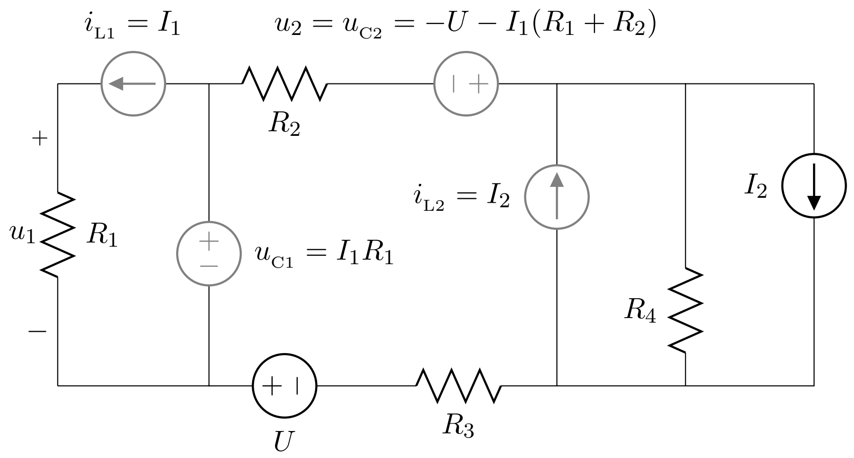
Instead, we take advantage of continuity. The capacitors' *voltages* and the inductors' *currents* cannot change instantaneously, so we can calculate what they were before the change, by analysing the circuit from part 'a'. Then we can put suitable sources into the circuit for part 'b' (below) that have these values, and solve for $t = 0^+$.

The following shows the circuit at $t = 0^-$ (as in Q4a) but with the continuous quantities defined: i_{L1}, i_{L2}, u_{C1}, u_{C2}.



These continuous quantities of capacitors and inductors can be found by basic KCL/KVL/Ohm. The hardest is u_{C2}, which is found from KVL around the loop of L_2 (a short circuit, as the circuit is in equilibrium), R_3 (also no voltage, seen by Ohm's law as it has no current), U, R_1, L_1 (short circuit as for L_2) and R_2. It could have been nicer to swap the direction in which u_{C2} is defined, to avoid the negative signs.

The circuit below is valid for $t = 0^+$. Inductors and capacitors are replaced with the appropriate sources (current and voltage respectively), with values equal to the continuous quantities found from the circuit above.



Then u_1 can be found directly by KCL/Ohm, as this resistor is in series with a current source.

$$u_1(0^+) = i_{L1} R_1 = I_1 R_1.$$

The energy W_{C2} can be written directly from the standard formula for energy in a capacitor,

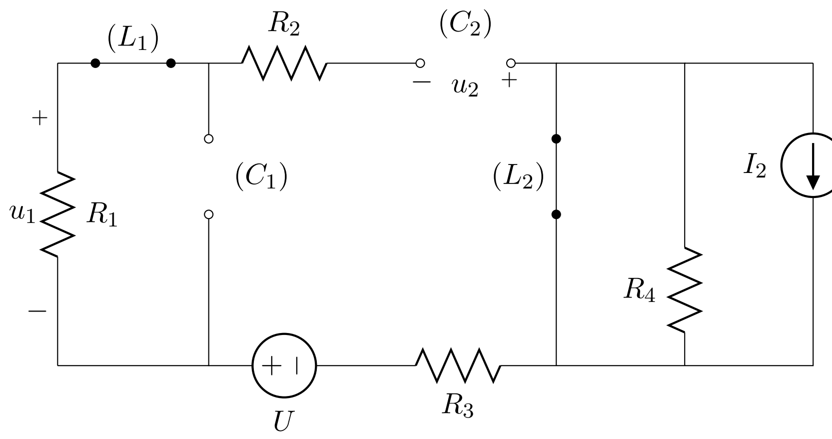
$$W_{C2}(0^+) \;=\; \frac{1}{2}\, C_2\, u_{C2}^2 \;=\; \frac{1}{2}\, C_2\; \left(U + I_1(R_1{+}R_2)\right)^2.$$

The power P_{R2} can be found from the current through or voltage across R_2. We can take KVL around a loop of resistors and voltage sources: U, u_{C1}, R_2, u_{C2}, R_4, R_3. KCL dictates that R_2 and R_3 have the same current. Notice that in the nodes that R_4 connects to, i_{L2} and I_2 cancel each other (equal, opposite): so the current in R_4 equals the current in R_3 and R_2. Defining this unknown current as i, clockwise, KVL becomes

$$U + (I_1 R_1) - iR_2 + (-U - I_1(R_1{+}R_2)) - iR_4 - iR_3 = 0 \qquad \Longrightarrow \qquad i = \frac{-I_1 R_2}{R_2 + R_3 + R_4}.$$

$$P_{\text{R2}}(0^+) \;=\; i^2 R_2 \;=\; \left(\frac{I_1 R_2}{R_2+R_3+R_4}\right)^2 R_2 \;=\; \frac{I_1^2 R_2^3}{(R_2+R_3+R_4)^2}.$$

c) This is again an equilibrium. Its difference from the previous case (part 'a' at $t=0^-$) is that the left current source now has zero current, due to the $1-\mathbf{1}(t)$ function.



In this circuit, with C_2 being an open circuit in the equilibrium, and therefore blocking any current around the loop with R_3, there can be no voltage across R_2 (Ohm's law if there's no current there) and no current passes through the voltage source or the resistors R_1 and R_2. All of the current I_2 will pass through the short-circuit of L_2.

By KVL around the loop of C_2 (unknown), L_2 (short), R_3 (no current), U, R_1 (no current), L_1 (short), R_2 (no current), we find $u_2 = 0+0-U+0+0+0 = -U$, so

$$u_2(\infty) = -U$$

Q5

Method 1: direct ODE. Define the voltage across the capacitor as u, in passive convention relative to i. At times $t>0$ the source at the right has value U. KCL at the node above the capacitor gives

$$i(t) + \frac{u(t)-Ku_x(t)}{R_1} + \frac{u(t)-U}{R_2} = 0.$$

Substituting $u_x(t) = U-u(t)$, and $i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t}$, this becomes an ODE in $u(t)$,

$$C\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{u(t)-K(U-u(t))}{R_1} + \frac{u(t)-U}{R_2} = 0$$

which in 'standard' form is

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \left(\frac{R_1+(1+K)R_2}{R_1R_2C}\right)u(t) = \frac{R_1+KR_2}{R_1R_2C}U.$$

This has the general solution

$$u(t) = \frac{R_1+KR_2}{R_1R_2}U + A\,\mathrm{e}^{-\frac{R_1+(1+K)R_2}{R_1R_2C}t},$$

where the constant A needs to be determined by knowledge of $u(t)$ at some specific time. In our case, $u(0^-)$ can be found as an equilibrium solution, then $u(0^+)$ will be the same, because the capacitor's voltage is a continuous quantity. (Be careful: if we were working with another quantity, such as the capacitor's current, there might be a discontinuity [jump] between 0^- and 0^+. What we actually want is the value at the start of the 'new' state of the circuit, after the step function: this is 0^+.)

At $t = 0^-$, the only independent source in the circuit has a value of zero. From this, we can argue that "with nothing to drive the circuit, all values will be zero": so $u(0^-) = 0$. (To be stricter about this, we might want to check whether the circuit is stable, i.e. whether the Thevenin source seen by the capacitor has positive resistance. This could depend on the actual value of K and other components.)

Putting in this known value of $u(0^+) = u(0^-) = 0$,

$$u(0) = 0 = \frac{R_1 + KR_2}{R_1 R_2} U + A\,\mathrm{e}^0 \qquad \Longrightarrow \qquad A = -\frac{R_1 + KR_2}{R_1 R_2} U.$$

from which the complete solution for $t > 0$ is

$$u(t) = \frac{R_1 + KR_2}{R_1 + (1{+}K)R_2}\, U \left(1 - \mathrm{e}^{-t/\frac{CR_1R_2}{R_1+(1+K)R_2}}\right).$$

We found $u(t)$ because it is the continuous variable, so its initial condition is therefore easily found from the equilibrium before the step.

But it was $i(t)$ that we were supposed to find! We have two equations (above) that relate $u(t)$ and $i(t)$: one is the first KCL we wrote, and the other is the capacitor-equation that we used to eliminate $i(t)$ from the KCL in order to get a simple ODE.

It's probably simplest to use the capacitor equation to find the current:

$$i(t) \;=\; C\frac{\mathrm{d}}{\mathrm{d}t}u(t) \;=\; C\,\frac{-(R_1 + KR_2)}{R_1 + (1{+}K)R_2}\, U \left(-\frac{R_1 + (1{+}K)R_2}{CR_1R_2}\right) \mathrm{e}^{-t/\frac{CR_1R_2}{R_1+(1+K)R_2}} \qquad (t > 0).$$

After tidying, we can answer with

$$i(t) = \frac{R_1 + KR_2}{R_1 R_2}\, U\, \mathrm{e}^{-t/\frac{CR_1R_2}{R_1+(1+K)R_2}} \qquad (t > 0).$$

Method 2: Initial, Final, Time-constant.

In this circuit, the topology doesn't change: only the value of the independent source changes.
If we start with KCL (as in Method 1), for $t > 0$,

$$i(t) + \frac{u(t) - Ku_x(t)}{R_1} + \frac{u(t) - U}{R_2} = 0.$$

but don't eliminate $i(t)$, we get the i, u relation at the terminals of the capacitor:

$$i(t) + \frac{u(t) - K(U - u(t))}{R_1} + \frac{u(t) - U}{R_2} = 0.$$

A Thevenin or Norton equivalent of the circuit (other than the capacitor) must also have this i, u relation.
Now we group the $u(t)$ and U terms,

$$i(t) = \frac{R_1 + KR_2}{R_1 R_2} U - \frac{R_1 + KR_2}{R_1 R_2} u(t)$$

and compare to the i, u relation for a two-terminal equivalent (for example a Norton source, $i = I_{\mathrm{N}} - \frac{u}{R_{\mathrm{N}}}$) in order to identify the equivalent-source quantities,

$$R_{\mathrm{N}} = \frac{R_1 R_2}{R_1 + KR_2}, \qquad I_{\mathrm{N}} = \frac{R_1 + KR_2}{R_1 R_2} U, \qquad U_{\mathrm{T}} = I_{\mathrm{N}} R_{\mathrm{N}} = U.$$

The above equivalent is valid for $t \geq 0$. For $t < 0$ the only difference is that we have 0 instead of U in this equation. This means that the Norton (or Thevenin) resistance is the same as above (R_{N}), but the source has zero value. The equivalent circuit is therefore just a resistor of R_{N}.

We could use this equivalent to find $u(t)$ and then derive $i(t)$ from that. Or we can try finding $i(t)$ directly: as this is not a continuous quantiy we must be careful to choose the initial value as $i(0^+)$, not $i(0-)$.

Let's find $i(t)$ directly, from the initial value, final value and time-constant.

The final value is $i(\infty) = 0$, as as capacitor in an equilibrium state behaves as an open circuit.

At $t = 0^-$ the equivalent source is just a resistor: the capacitor's voltage is $u(0^-) = 0$ as this is the equilibrium state of a capacitor connected to a resistor. By continuity, $u(0^+) = u(0^-) = 0$. The initial current $i(0^+)$ is therefore the short-circuit current of the equivalent source, $i(0^+) = \frac{R_1+KR_2}{R_1R_2}U$, since the source at that time has a zero voltage (the capacitor) at its terminals.

The time-constant is $\tau = CR_{\rm N} = \frac{CR_1R_2}{R_1+KR_2}$.

Putting all the above together,

$$i(t) = i(\infty) + \big(i(0^+) - i(\infty)\big)\, \mathrm{e}^{-t/\tau} = \frac{R_1 + KR_2}{R_1R_2} U\, \mathrm{e}^{-t\big/\frac{CR_1R_2}{R_1+(1+K)R_2}} \qquad (t > 0).$$

Q6

Let's use peak-value and sine reference: time-function $U(t) = \hat{U}\sin(\omega t)$ becomes phasor $U(\omega) = \hat{U}\underline{/0}$. We will work with phasors (frequency-domain) until expressing the final answers in time-function form.

a) With $M = 0$, KVL around the left loop gives

$$U(\omega) = Ri(\omega) + \mathrm{j}\omega L_1 i(\omega),$$

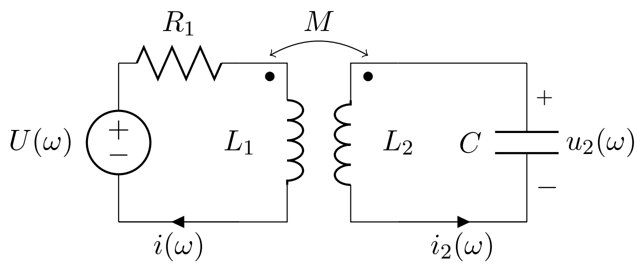
so

$$i(\omega) = \frac{U(\omega)}{R + \mathrm{j}\omega L_1}.$$

Converting this back to a time-function, using the same choice of reference as when converting from time to frequency,

$$i(t) = \frac{\hat{U}}{\sqrt{R^2 + \omega^2 L_1^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega L_1}{R}\right).$$

b) With $M \neq 0$, it's more complicated! Writing the whole mutual-inductor equation, and defining a current and voltage on the right-hand side of the inductors,



$$\begin{aligned} \text{left loop:}\quad U &= Ri + \mathrm{j}\omega L_1 i + \mathrm{j}\omega M i_2 \\ \text{right loop:}\quad u_2 &= \mathrm{j}\omega L_2 i_2 + \mathrm{j}\omega M i. \end{aligned}$$

These are two equations in three unknowns: we need to substitute the relation of current and voltage for the capacitor, which in view of the directions of i_2 and u_2 is $u_2 = \frac{-i_2}{\mathrm{j}\omega C}$; this gives

$$\begin{aligned} U &= Ri + \mathrm{j}\omega L_1 i + \mathrm{j}\omega M i_2 \\ \frac{-i_2}{\mathrm{j}\omega C} &= \mathrm{j}\omega L_2 i_2 + \mathrm{j}\omega M i. \end{aligned}$$

Rearranging the second equation in terms of i_2 gives

$$\frac{-i_2}{\mathrm{j}\omega C} = \mathrm{j}\omega L_2 i_2 + \mathrm{j}\omega M i \quad\Longrightarrow\quad \frac{i_2}{\omega C} = \omega L_2 i_2 + \omega M i \quad\Longrightarrow\quad i_2 = \frac{\omega M}{\frac{1}{\omega C} - \omega L_2} i$$

which we substitute into the first equation to give

$$U = (R + \mathrm{j}\omega L_1)\, i + \mathrm{j}\omega M \frac{\omega M}{\frac{1}{\omega C} - \omega L_2} i$$

This gives the sought i as

$$i = \frac{U}{R + \mathrm{j}\omega\left(L_1 + \frac{\omega M^2}{\frac{1}{\omega C} - \omega L_2}\right)}$$

from which the time-function can be found as in part 'a', but with a less elegant imaginary part!

$$i = \frac{U}{\sqrt{R^2 + \left(\omega L_1 + \frac{\omega^2 M^2}{\frac{1}{\omega C} - \omega L_2}\right)^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega L_1 + \frac{\omega^2 M^2}{\frac{1}{\omega C} - \omega L_2}}{R}\right).$$

Q7

a) This circuit is a basic inverting amplifier, except that the two impedances (in the input and in the feedback path) are each made up of several components. Based on the usual result from KCL, for an inverting amplifier,

$$H(\omega) \;=\; -\frac{Z_{\text{feedback}}}{Z_{\text{input}}} \;=\; -\frac{\frac{R_2 \frac{1}{\mathrm{j}\omega C_2}}{R_2 + \frac{1}{\mathrm{j}\omega C_2}}}{R_0 + \frac{R_1 \frac{1}{\mathrm{j}\omega C_1}}{R_1 + \frac{1}{\mathrm{j}\omega C_1}}} \;=\; -\frac{\frac{R_2}{1+\mathrm{j}\omega C_2 R_2}}{R_0 + \frac{R_1}{1+\mathrm{j}\omega C_1 R_1}} \;=\; -\frac{\frac{R_2}{1+\mathrm{j}\omega C_2 R_2}}{\frac{R_0(1+\mathrm{j}\omega C_1 R_1)+R_1}{1+\mathrm{j}\omega C_1 R_1}}$$

Collecting the terms into a single numerator and denominator,

$$H(\omega) \;=\; -\frac{R_2(1+\mathrm{j}\omega C_1 R_1)}{(R_0(1+\mathrm{j}\omega C_1 R_1)+R_1)(1+\mathrm{j}\omega C_2 R_2)} \;=\; -\frac{R_2(1+\mathrm{j}\omega C_1 R_1)}{(R_0+R_1+\mathrm{j}\omega C_1 R_0 R_1)(1+\mathrm{j}\omega C_2 R_2)}$$

which can then be put in a more 'canonical' (standard) form,

$$H(\omega) \;=\; \frac{-R_2(1+\mathrm{j}\omega C_1 R_1)}{(R_0+R_1)(1+\mathrm{j}\omega C_1\frac{R_0 R_1}{R_0+R_1})(1+\mathrm{j}\omega C_2 R_2)}.$$

b) The above result for $H(\omega)$ can be expressed in the requested form,

$$H(\omega) = \frac{-K\,(1+\mathrm{j}\omega/\omega_a)}{(1+\mathrm{j}\omega/\omega_b)(1+\mathrm{j}\omega/\omega_c)},$$

if we choose to define: $K = \frac{R_2}{R_0+R_1}$, $\omega_a = \frac{1}{C_1 R_1}$, $\omega_b = \frac{1}{C_2 R_2}$, $\omega_c = \frac{R_0+R_1}{C_1 R_0 R_1}$.
(The choice of ω_b and ω_c can be swapped, as the terms containing this parameters are interchangeable in the requested form of the equation.)

Part 'a' could have been answered with an expression of very different structure from the one shown in our solution; in that case, further rearrangement may be necessary in order to see how the parameters ω_a etc relate to the circuit components' values.

c) The plot of $|H(\omega)|$ is shown below. As a Bode-plot was requested, the vertical scale should be in dB, and the horizontal scale of frequency should be logarithmic. The straight black lines in this plot show the usual 'asymptotic approximation', and the dashed blue line shows the actual function.

The chosen numbers on the frequency scale are arbitrary units, but note that the question did require that $\omega_b = 100\,\omega_a$.

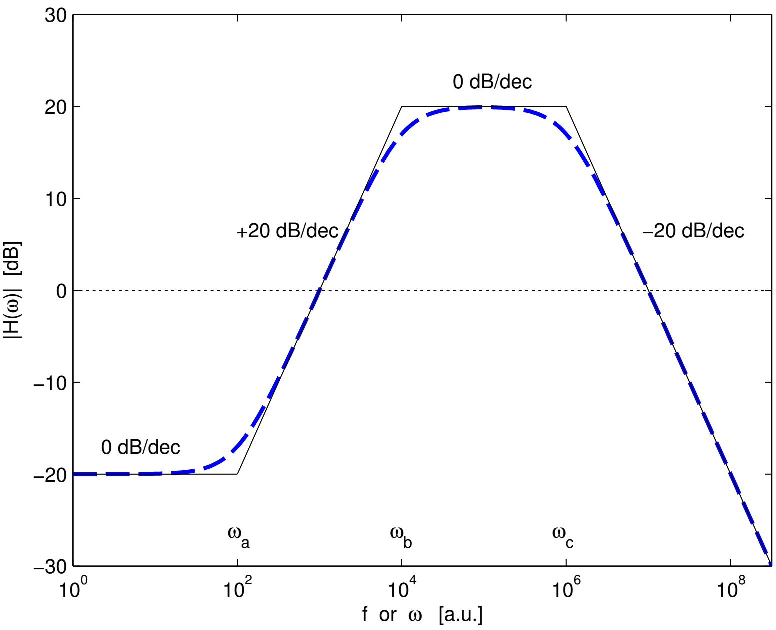
We have also chosen $\omega_c = 100\,\omega_b$, although the only requirement was $\omega_c => \omega_b$. If we chose $\omega_c = 10\,\omega_b$, then the flat part at the top would be only half as long, and the blue curve (the actual function instead of the asymptotic approximation) would hardly show any flatness.

At $\omega \ll \omega_a$, we have just $|H(\omega)| = K$.
The question specifies that $K = 0.1$, so the level is $20\,\log_{10} 0.1\,\mathrm{dB}$, which is $-20\,\mathrm{dB}$.

When $\omega_a < \omega < \omega_b$, the magnitude of the $(1 + j\omega/\omega_a)$ term rises at 20 dB/decade, but the other two terms still have magnitude of 0 dB. From this, we know that when ω_b is reached (which is *two* decades higher, because $\omega_b = 100\,\omega_a$) the level is $-20\,\text{dB} + 2 \cdot (20\,\text{dB}) = +20\,\text{dB}$.

When ω exceeds ω_b, the $\frac{1}{1+j\omega/\omega_b}$ term decreases in magnitude at -20 dB/decade, cancelling the change that the $(1 + j\omega/\omega_a)$ would cause, so the plot stays flat.

Above ω_c, there is an additional decreasing magnitude from $\frac{1}{1+j\omega/\omega_c}$, so the function decreases.



Q8

a) It should be an inductor. (No explanation was demanded: this was an 'easy point'.) When a maximum power is to be obtained, by connecting a load impedance Z_l to a source with some impedance Z_s, the ac maximum power criterion $Z_l = Z_s^*$ tells us that the reactive part in one of these impedances (a capacitor in the source in our case) requires that the opposite type of reactive component in the other impedance (an inductor in the load in our case). One could alternatively show this from the equations that are derived in Q8b.

b) $L = \dfrac{C}{\frac{n^2}{R_1^2} + n^2\omega^2C^2} = \dfrac{CR_1^2}{n^2\left(1 + \omega^2C^2R_1^2\right)}$.

The criterion is $Z_l = Z_s^*$, as mentioned above. We have some flexibility in defining our load and source! Clearly R_2 is part of the load, since this is where we want our 'maximum power' to be transferred to. We could define all the other components as the source. Or we could include X as part of the load, as was implied in our answer to Q8a. Or we could include X and the transformer as the load, etc! All these options will give the same result.

Let's consider the load to be R_2 and X: we define $Z_l = R_2 + Z_x$, where Z_x is the impedance of component X. The source is then the remaining part of the circuit. We do not need to find the power, but just the condition on Z_l to obtain maximum power at the load. Thus, only the source's impedance is needed, not its Thevenin voltage or Norton current. Seen at the right-hand-side terminals of the transformer, the source impedance is the parallel combination of R_1 and C, scaled by $1/n^2$,

$$Z_s = \frac{R_1 \frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}} \cdot \frac{1}{n^2} = \frac{R_1/n^2}{1 + j\omega CR_1} = \frac{R_1\left(1 - j\omega CR_1\right)}{n^2\left(1 + \omega^2C^2R_1^2\right)}$$

Using the above results, and the condition for ac maximum power,

$$Z_l = R_2 + Z_x = Z_s^* = \frac{R_1\left(1 + j\omega C R_1\right)}{n^2\left(1 + \omega^2 C^2 R_1^2\right)},$$

from which 'equating the imaginary parts', assuming Z_x to be purely imaginary, shows that

$$Z_x = \frac{j\omega C R_1^2}{n^2\left(1 + \omega^2 C^2 R_1^2\right)}.$$

The single component that has a positive, imaginary impedance is an inductor. If we treat X as an inductor, L, and substitute $Z_x = j\omega L$, the required value is

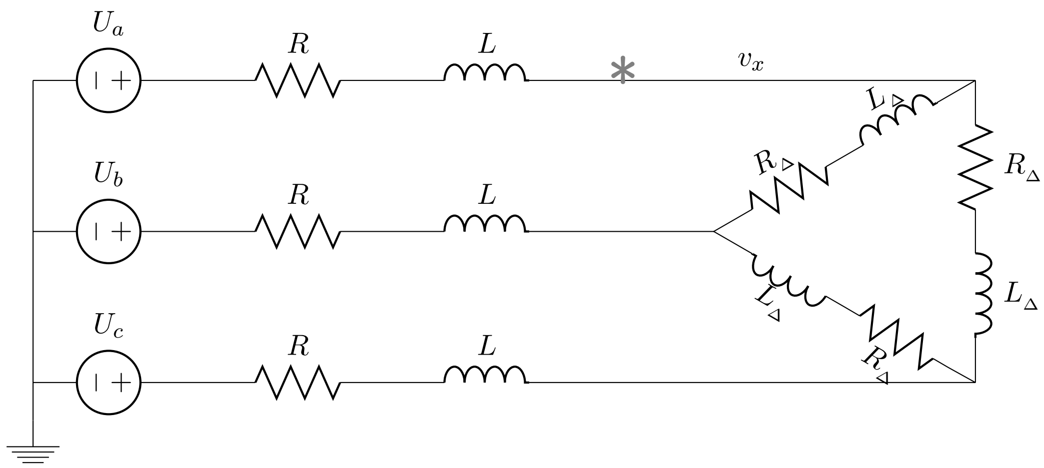
$$L = \frac{C R_1^2}{n^2\left(1 + \omega^2 C^2 R_1^2\right)}.$$

Question: why did we assume that Z_x was purely imaginary? The question says that it must be one of a capacitor, inductor or resistor. If it is a resistor, then we cannot fulfill the maximum power transfer condition, as we can only make the load have a real value of impedance. (Also, adding resistance outside the component where maximum power is wanted is not helpful; it just reduces the current in the circuit.) So we assume a capacitor or inductor, either of which is a purely imaginary impedance.

c) $R_2 = \dfrac{R_1}{n^2\left(1 + \omega^2 C^2 R_1^2\right)}.$

This can be found from the expression for Z_l in Q8b, by equating the real parts, again with the assumption that Z_x is purely imaginary.

Q9



a) With $R = 0$ and $L = 0$, the source is directly connected to the load, so we know that the line-voltage at the load-terminals is U.

Using the standard result for complex power into a balanced Δ-connected impedance load with balanced supply of line-voltage U,

$$S = \frac{3U^2}{\left(R_\Delta + j\omega L_\Delta\right)^*}.$$

It was the apparent power that we were supposed to find:

$$|S| = \frac{3U^2}{|R_\Delta + j\omega L_\Delta|} = \frac{3U^2}{\sqrt{R_\Delta^2 + \omega^2 L_\Delta^2}}.$$

The power factor is PF $= P/|S|$. We already have found apparent power $|S|$. To find active power P,

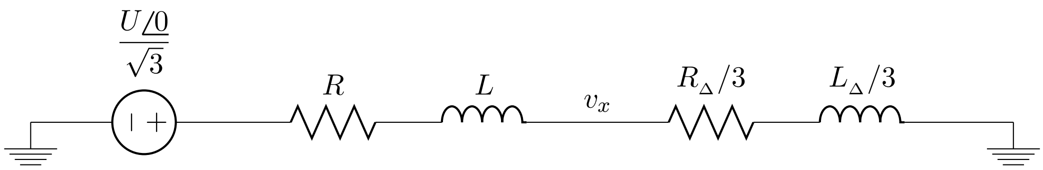
$$S = P + \mathrm{j}Q = \frac{3U^2}{(R_\Delta + \mathrm{j}\omega L_\Delta)^*} = 3U^2 \frac{R_\Delta + \mathrm{j}\omega L_\Delta}{R_\Delta^2 + \omega^2 L_\Delta^2} \qquad \Longrightarrow \qquad P = \frac{3U^2 R_\Delta}{R_\Delta^2 + \omega^2 L_\Delta^2},$$

giving

$$\text{PF} = \frac{\frac{3U^2 R_\Delta}{R_\Delta^2 + \omega^2 L_\Delta^2}}{\frac{3U^2}{\sqrt{R_\Delta^2 + \omega^2 L_\Delta^2}}} = \frac{R_\Delta}{\sqrt{R_\Delta^2 + \omega^2 L_\Delta^2}}$$

If you remember the first exercises from topic 11 (ac power: calculating PF, S, etc for series and parallel components with voltage or current sources feeding them), you might have reached this expression for the power factor without using intermediate steps of P, $3U$ etc. In fact, we can calculate it by looking just at the R and ωL values of the series branches. This is most easily seen if defining a particular current through the branch: then the active power is $|i|^2 R_\Delta$, and the apparent power is $|i|^2 |R_\Delta + \mathrm{j}\omega L_\Delta|$.

b) A single-phase equivalent of this balanced circuit is the following. The source represents one phase of a Y-connected source, so it has magnitude of $1/\sqrt{3}$ of the line-voltage.



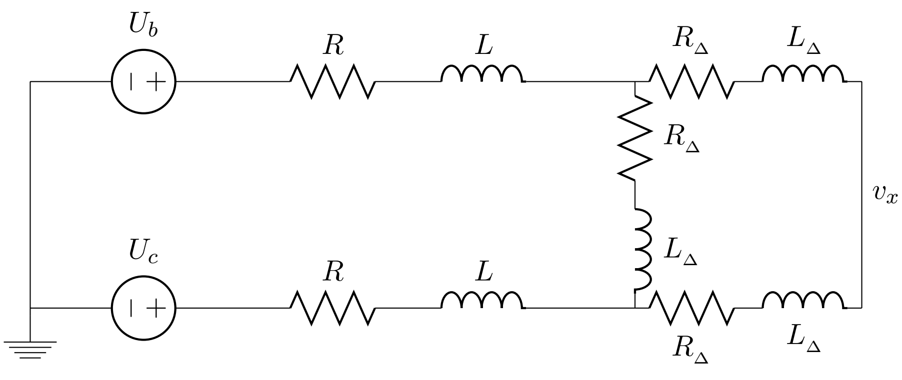
Here, the choice of zero phase at the source means that it fits with what the question stated about phase a: its potential has been defined as having zero phase. By voltage divison between the line and load impedances, we find the phasor v_x.

$$v_x = \frac{U}{\sqrt{3}} \cdot \frac{R_\Delta/3 + \mathrm{j}\omega L_\Delta/3}{R + R_\Delta/3 + \mathrm{j}\omega(L + L_\Delta/3)} = \frac{U}{\sqrt{3}} \cdot \frac{R_\Delta + \mathrm{j}\omega L_\Delta}{3R + R_\Delta + \mathrm{j}\omega(3L + L_\Delta)}$$

This could arguably be improved by multiplying up and down by the complex conjugate of the denominator, to get it in purely $\Re + \mathrm{j}\Im$ form ... but then again, that could arguably be considered messy!

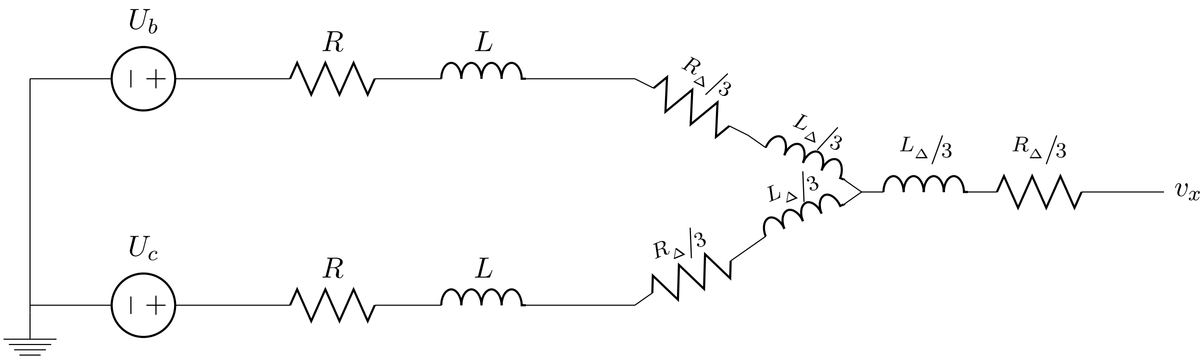
c) $|v_x| = \dfrac{U}{2\sqrt{3}}$.

The break (open-circuit) disconnects source 'U_a'.



This could look a bit complicated ... it has two parallel branches, and we want to find the potential at the middle of one of them.

A Δ-Y transform of the load helps make it clearer.



There is no current through the branch at the far right, so by Ohm's law the potential at the star-point (centre of the Y-load) is the same as v_x. Thus, by KCL at the star-point,

$$\frac{U_b - v_x}{R+\mathrm{j}\omega L + {}^{R_\Delta}/3 + {}^{L_\Delta}/3} + \frac{U_c - v_x}{R+\mathrm{j}\omega L + {}^{R_\Delta}/3 + {}^{L_\Delta}/3} = 0 \qquad \Longrightarrow \qquad v_x = \frac{U_b + U_c}{2}.$$

Putting in the known values of the source voltages,

$$v_x = \frac{U}{2\sqrt{3}}\left(1\angle{-{}^{2\pi}/3} + 1\angle{+{}^{2\pi}/3}\right) = \frac{-U}{2\sqrt{3}} = \frac{U}{2\sqrt{3}}\angle\pi.$$

Only the magnitude was required by the question. Notice that if one is very confident with phasor diagrams and voltage division, this result might be convincing enough without the algebra, by taking the mid-point of a line drawn between the U_b and U_c phasors.