KTH EI1120 Elkretsanalys (CENMI) Omtenta 2017-06-08 kl 14–19

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Upp till tre A4-ark (sex sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek. De måste *inte* lämnas in.

För den intresserade: tre ark för att man kan välja att återanvända vad man hade till KS1 och KS2, samt att lägga till en ny för växelström.

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. *Senare deltal brukar vara svårare* att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt A, B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. A=12, B=10, C=18. Låt a, b och c vara poängen man får i dessa respektive delar i tentan, och a_{k} vara poängen man fick från kontrollskrivning KS1, och b_{k} poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a, a_{\mathrm{k}})}{A} \geq 0{,}4 \quad \& \quad \frac{\max(b, b_{\mathrm{k}})}{B} \geq 0{,}4 \quad \& \quad \frac{c}{C} \geq 0{,}3 \quad \& \quad \frac{\max(a, a_{\mathrm{k}}) + \max(b, b_{\mathrm{k}}) + c + h}{A + B + C} \geq 0{,}5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Nathaniel Taylor (0739 498 572)

Del A. Likström

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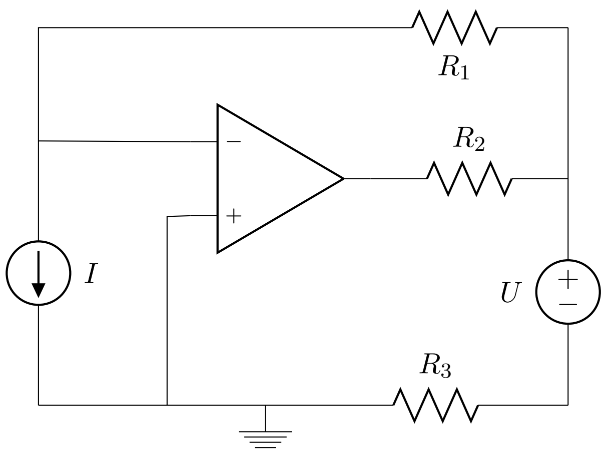
1) [4p] Bestäm följande:

a) [1p] Effekten absorberad av R_1.

b) [1p] Effekten levererad av källan I.

c) [1p] Effekten absorberad av R_3.

d) [1p] Effekten levererad av operationsförstärkaren.

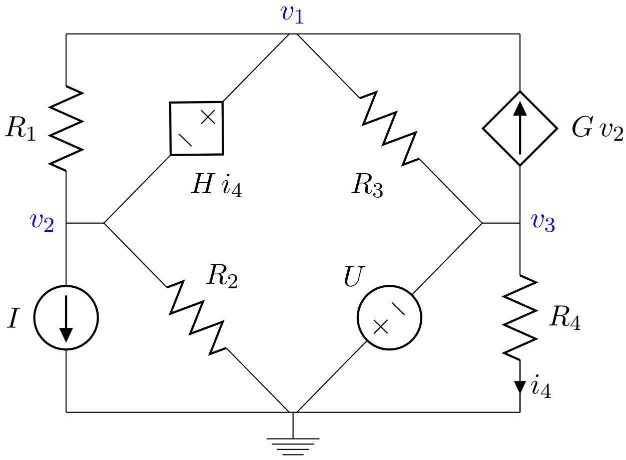


2) [4p]

Bestäm v_1 genom valfri metod; alternativt, använd nodanalys för att skriva ekvationer som skulle kunna lösas för att bestämma v_1.

Hela poäng erhålls vid korrekt lösning av v_1. Vid korrekta ekvationer utan hela lösningen av v_1 erhålls 0,5p mindre.

Observera att du kan börja med att använda nodanalys för att få ekvationerna, och sedan försöka lösa dessa för full poäng. Då får du en trygghet i att ekvationerna ger flest poäng även om du inte lyckas lösa dem.

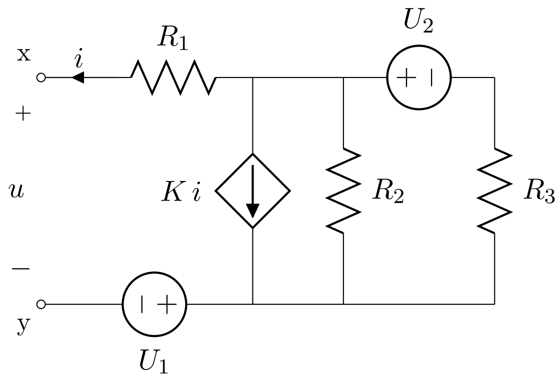


3) [4p]

a) [3p] Vilket värde av motstånd måste kopplas mellan kretsens poler x-y, om den maximala möjliga effekten ska dras från kretsen?

b) [1p] Hur mycket är den maximala effekten som kretsen kan leverera?

För dessa svar (3a, 3b) krävs inte att man förenklar uttrycken: de kan vara svåra att förenkla, beroende på vilken lösningssätt som man använt. Visa tydligt din lösningsmetod.



Short translations of Section-A questions to English:

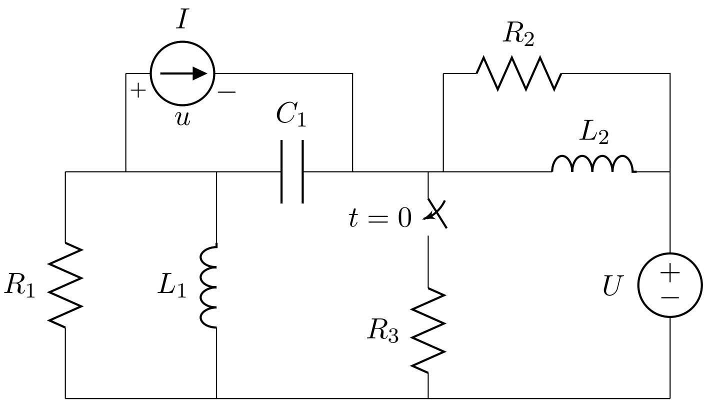
1. Find: a) power into R_1, b) power from I, c) power into R_3, d) power from opamp.

2. Two *alternative* options: [for all 4 points] solve for v_1 completely, by any method you wish; or [for 3.5 points] use nodal analysis to write (but not have to solve) a set of equations that could be solved to find v_1.

3. A maximum power and two-terminal equivalent question.
a) A resistor R_x is connected at the terminals x-y: what value should R_x be in order to obtain the maximum possible power from the shown circuit?
b) How much is this maximum power that the circuit can deliver?
(Don't worry about simplifying your answers. Just show the method.)

Del B. Transient

4) [5p] Bestäm följande:

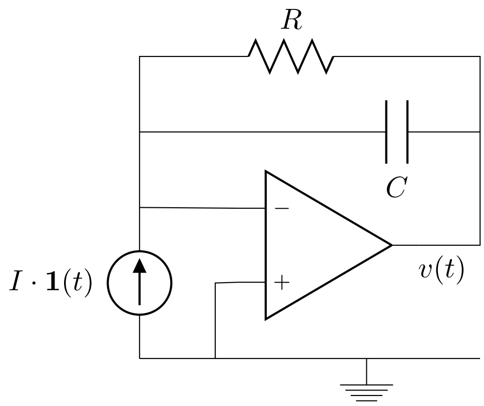


a) [2p] Vid $t = 0^{-}$:
energin lagrad i kondensatorn C_1,
effekten levererad av spänningskällan U.

b) [2p] Vid $t = 0^{+}$:
spänningen u över strömkällan,
effekten absorberad i motståndet R_3.

c) [1p] Vid $t \to \infty$:
effekten levererad av strömkällan I.

5) [5p]



Bestäm $v(t)$, för $t > 0$.

Obs: $\mathbf{1}(t)$ är enhetsstegfunktionen.

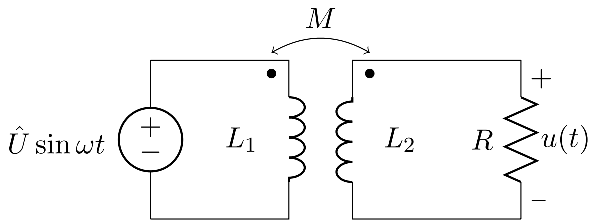
Short translations of Section-B questions to English:

4. Find the following quantities:
a) at $t = 0^{-}$: energy stored in C_1, power out of source U.
b) at $t = 0^{+}$: voltage u, power into R_3.
c) as $t \to \infty$: power out of source I.

5. Determine $v(t)$, for $t > 0$.

Del C. Växelström

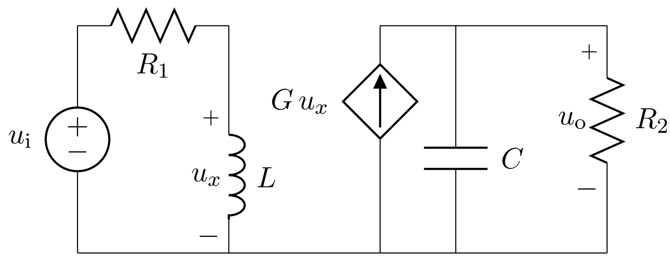
6) [4p]



Kretsen innehåller två kopplade spolar, med ömsesidig induktans M.

Bestäm $u(t)$.

7) [5p]



a) [2p] Bestäm kretsens nätverksfunktion,

$$H(\omega) = \frac{u_\mathrm{o}(\omega)}{u_\mathrm{i}(\omega)}.$$

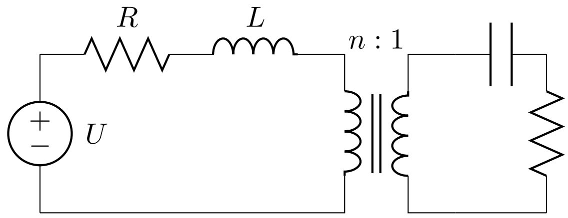
b) [1p]

Visa att svaret till deltal 'a' kan skrivas i den följande formen,

$$H(\omega) = \frac{k\ \mathrm{j}\omega/\omega_a}{(1+\mathrm{j}\omega/\omega_a)\,(1+\mathrm{j}\omega/\omega_b)}.$$

c) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'.
Anta att $\omega_b = 1000\,\omega_a$, och $k = 10$.
Markera kända frekvenser (t.ex. ω_1), nivåer (t.ex. x dB) och lutningar (t.ex. y dB/dekad).

8) [3p]

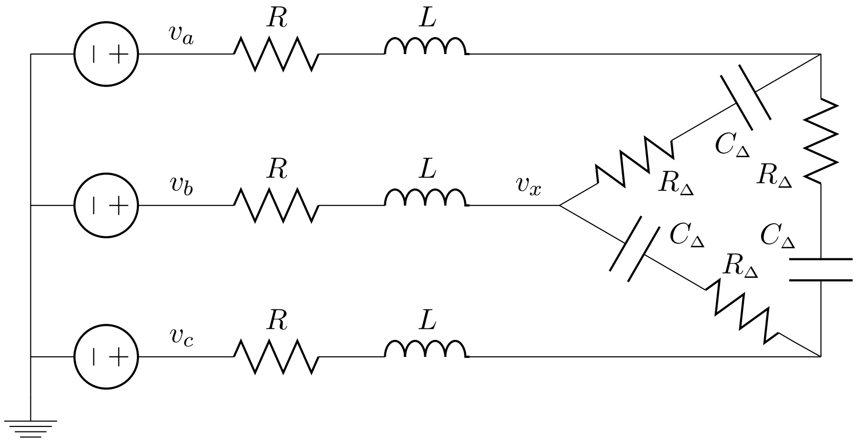


Källan är en växelspänningskälla med vinkelfrekvens ω. Värden U, ω, R, L och n är fastställda, kända storheter. Kondensatorn och motståndet på transformatorns högersida har värden som är anpassad för att maximera den effekt som levereras till detta motstånd (på höger sidan). Bestäm denna effekt.

9) [6p]

Spänningskällorna utgör en balanserad trefaskälla med huvudspänning U och vinkelfrekvens ω.

Komponenterna R_Δ och C_Δ modellerar faserna av en last, och komponenterna R och L modellerar fasledarna mellan källan och lasten.



a) [2p] Bestäm den aktiva effekten försörjd av den trefas källan.

b) [2p] Vilket värde ska C_Δ ha för att den trefas källan matar med 'perfekt' effektfaktor (PF = 1). Uttrycka värdet i förhållande till de andra komponentvärden.

c) [2p] Bestäm potentialen v_x. Obs: inte bara en magnitud, utan ett komplext värde (fasvektor). Anta att $\angle v_a = 0$ och fasföljden ('phase-rotation') är abc.

Short translations of Section-C questions to English:

6. The circuit has two inductors, between which there is mutual inductance M.
Determine $u(t)$.

7. Two first-order filters, cascaded via a dependent source.
a) determine the function $H(\omega) = u_o(\omega) / u_i(\omega)$.
b) show that the above function can be written as $H(\omega) = \frac{k\, j\omega/\omega_a}{(1+j\omega/\omega_a)\,(1+j\omega/\omega_b)}$.
c) Sketch a Bode amplitude-plot of the function from part 'b'.
Assume $1000\,\omega_a = \omega_b$ and $k = 10$.
Indicate known values of frequencies (e.g. ω_1), levels (e.g. x dB) and gradients (e.g. y dB/decade).

8. The source is an ac voltage-source with angular frequency ω.
The marked component-values are fixed. The values of the capacitor and resistor on the right of the transformer have been chosen to obtain maximum power transfer into that resistor (on the right). Determine this power (into the resistor on the right).

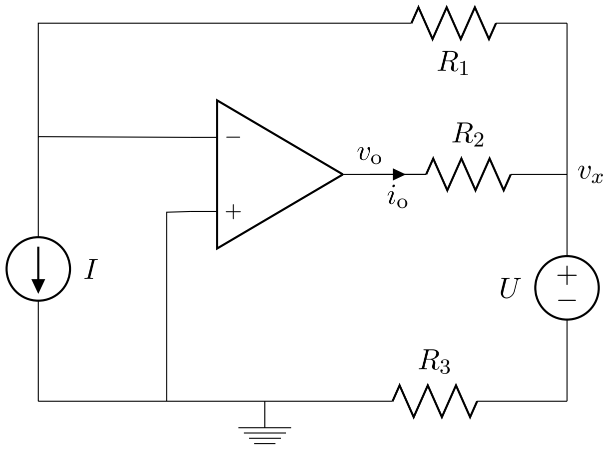
9. The sources form a balanced three-phase source with line-voltage magnitude U and angular frequency ω.
Each $R\,L$ branch represents one phase of a power line; each $R_\Delta\,C_\Delta$ branch represents one phase of a load.
a) Find the active power delivered by the three-phase source.
b) What value should C_Δ have in order that the three-phase source supplies power at a unity power-factor (PF= 1)? (Suggestion: write an equation for it in relation to the other component values, e.g. L.)
c) Find potential v_x (as a phasor, not just a magnitude), assuming that $\angle v_a = 0$ and phase-rotation is abc.

Slut … *men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren!*

Solutions (EI1120 TEN1 VT17, 2017-06-08)

Q1

We can start by defining some of the unknown quantities, for use in the solutions.



a) $P_{\mathrm{R1}} = I^2 R_1$.
No current flows in the ideal opamp inputs, so KCL requires that all the current I passes through R_1.

b) $P_{\mathrm{I}} = 0$.
The current source is connected between the nodes to which the opamp inputs are connected. The (ideal) opamp, with negative feedback, has equal potential at its two inputs. There is therefore no voltage across the current source I, so it delivers or consumes no power.

c) $P_{\mathrm{R3}} = \frac{(IR_1 - U)^2}{R_3}$.
By KVL, $v_x = 0 + IR_1$. This follows from comments in the previous two subquestions: the inverting input's potential is zero, and the current I flows in R_1. The voltage across R_3 is $v_x - U$, which is $IR_1 - U$. Power into a resistor R with voltage u across it is u^2/R, so in this case the power into R_3 is $(IR_1 - U)^2/R_3$.

d) $P_{\mathrm{op}} = I\left(I + \frac{IR_1 - U}{R_3}\right)R_1 + \left(I + \frac{IR_1 - U}{R_3}\right)^2 R_2$.
This is the product of the marked output potential and current, v_o and i_o. (Remember that we see the opamp as being a voltage source connected between the zero node and the output.)
From the earlier subquestions we know the current in R_1 and can calculate the current in R_3; KCL then gives the current in R_2, which is the opamp's output current, $i_o = I + \frac{IR_1 - U}{R_3}$.
By Ohm's law and KVL, $v_o = v_x + i_o R_2 = IR_1 + \left(I + \frac{IR_1 - U}{R_3}\right) R_2$.
Thus, $P_{\mathrm{op}} = v_o i_o = I\left(I + \frac{IR_1 - U}{R_3}\right)R_1 + \left(I + \frac{IR_1 - U}{R_3}\right)^2 R_2$.

Q2

Method 1: extended nodal analysis.

Define currents i_α and i_β in the two voltage sources (the dependent and independent voltage sources, respectively), into the + terminals. Then write KCL at each node except the reference (zero / ground / earth) node.

$$\begin{aligned}
\text{KCL}(1): \quad 0 &= \frac{v_1 - v_2}{R_1} + i_\alpha + \frac{v_1 - v_3}{R_3} - G v_2 && (1)\\
\text{KCL}(2): \quad 0 &= I + \frac{v_2 - v_1}{R_1} - i_\alpha + \frac{v_2}{R_2} && (2)\\
\text{KCL}(3): \quad 0 &= -i_\beta + \frac{v_3 - v_1}{R_3} + G v_2 + \frac{v_3}{R_4} && (3)
\end{aligned}$$

The voltage sources provide further relations, as well as the further unknowns.

$$\begin{aligned}
\text{source } U: \quad v_3 &= -U && (4)\\
\text{source } H\, i_4: \quad v_1 - v_2 &= H\, i_4 && (5)
\end{aligned}$$

We must define the marked quantities that are used as 'controlling variables'. In this circuit one of the dependent sources is controlled by a potential (v_2) that is already being used in the equations, so we do not need to define that.

$$i_4 = \frac{v_3}{R_4} \qquad (6)$$

Method 2: nodal analysis with supernodes etc.

There are four nodes. They can be treated as two supernodes: the nodes marked v_1 and v_2 are connected by the dependent voltage source, and the node marked v_3 is connected by the independent voltaage source to the reference node.
We do not need a KCL for the reference node or for any part of a supernode that contains it. So we only need one KCL equation, for the supernode of nodes 1 and 2.

When writing this, we can try to make substitutions to avoid getting extra unknown variables beyond v_1 into the equation.
The potential v_3 is seen to be $-U$.
Instead of $H\, i_4$, we can substitute $H\frac{-U}{R_4}$.
Instead of v_2, we write $v_1 - H\, i_4$, which becomes $v_1 + \frac{H}{R_4}U$.

$$\text{KCL(1,2)}: \quad 0 = I + \frac{v_1 + U}{R_3} + \left(v_1 + \frac{H}{R_4}U\right)\left(\frac{1}{R_2} - G\right)$$

This could alternatively have been found by substituting the above set of equations for the extended nodal analysis method into each other to eliminate all unknowns except v_1.

Rearranging to obtain v_1,

$$v_1 = \frac{\left(\frac{HG}{R_4} - \frac{H}{R_2 R_4} - \frac{1}{R_3}\right)U - I}{\frac{1}{R_2} + \frac{1}{R_3} - G}.$$

Q3

The two subquestions can be answered easily if we know the Thevenin (or Norton) equivalent of the circuit. If the Thevenin equivalent is an open circuit voltage U_{T} and resistance R_{T}, then:

a) Maximum power to load R_x requires $R_x = R_{\mathrm{T}}$.

b) Power transferred to load in this case is $P_{\max} = R_{\mathrm{T}} \left(\frac{U_{\mathrm{T}}}{2R_{\mathrm{T}}}\right)^2 = \frac{U_{\mathrm{T}}^2}{4R_{\mathrm{T}}}$.

How to find U_{T} and R_{T} is the more difficult part. Two methods are presented below. If there had not been a dependent source, the equivalent resistance could quickly have been found by setting all the sources to zero and simplifying the resulting resistor-network.

One method: short-circuit current and open-circuit voltage

The open-circuit value of terminal voltage u is relatively easily found. In open-circuit conditions, $i = 0$, so the dependent current source $K\,i$ also has zero current. Voltage division around the right-most loop gives the voltage across R_2. Then KVL around U_1, R_2, R_1 gives

$$u_{(\mathrm{oc})} = U_1 + U_2 \frac{R_2}{R_2 + R_3}.$$

The short-circuit value of terminal current i needs a bit more work. One method is nodal analysis, taking KCL on the node above R_2 (treating the node below R_2 as the reference). With the terminals short-circuited, there are four parallel branches. We can define a potential v at the node above R_2, and then substitute $v = iR_1 - U_1$; or we can do that substitution at the start, so that the equation can be written immediately in terms of i. The result is

$$i_{(\mathrm{sc})} = \frac{U_1 \left(\frac{1}{R_2} + \frac{1}{R_3}\right) + U_2 \frac{1}{R_3}}{1 + K + \frac{R_1}{R_2} + \frac{R_1}{R_3}}$$

By the usual relations of Thevenin parameters and open/short circuit values,

$$U_{\mathrm{T}} = u_{(\mathrm{oc})} \quad \text{and} \quad R_{\mathrm{T}} = \frac{u_{(\mathrm{oc})}}{i_{(\mathrm{sc})}}.$$

Another method: Derive the voltage/current relation

Direct expression for the u,i relation. This is quite similar to how the short-circuit current was derived, but now we take the case where neither u nor i is known. From KCL above R_2, with the voltage across R_2 defined as v,

$$\frac{v - u + U_1}{R_1} + K\,i + \frac{v}{R_2} + \frac{v - U_2}{R_3} = 0.$$

Substituting $i = \frac{v-u+U_1}{R_1}$, which is from KVL in the left branch,

$$(1+K)\,i + \frac{iR_1 + u - U_1}{R_2} + \frac{iR_1 + u - U_1 - U_2}{R_3} = 0$$

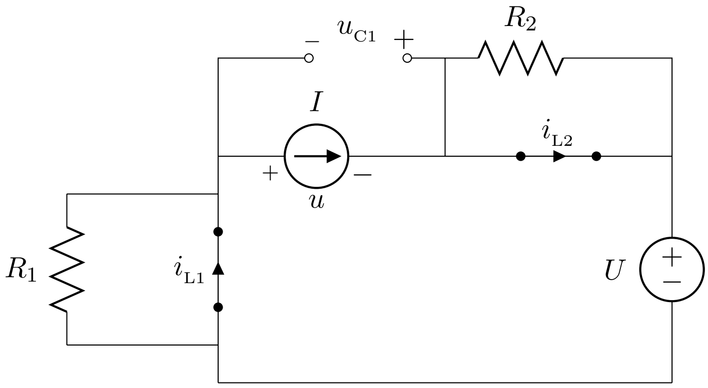
This equation is in terms of just u and i. We expect two unknowns for the one equation, as the system isn't fully defined: we haven't declared what is connected to the terminals.
Rearranging into the form $u = U_{\mathrm{T}} - iR_{\mathrm{T}}$,

$$u = \frac{\frac{U_1}{R_2} + \frac{U_1}{R_3} + \frac{U_2}{R_3} - i\left(1 + K + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right)}{\left(\frac{1}{R_2} + \frac{1}{R_3}\right)}.$$

Q4

a) $W_{\mathrm{C1}}(0^-) = \frac{1}{2}C_1U^2, \quad P_{\mathrm{U}}(0^-) = -UI.$

At $t=0^-$ the circuit can be redrawn in the following way. The switch is open, so its branch including R_3 does not affect the rest of the circuit. Capacitors and inductors are replaced by open- and short-circuits respectively, corresponding to their behaviour in the equilibrium state that is assumed before the switch changes the circuit.



The energy stored in the capacitor is $\frac{1}{2}C_1u_{\mathrm{C1}}^2$.
By KVL around the inductors (short-circuit) and voltage source, this is $u_{\mathrm{C1}}(0^-) = U$.
So $W_{\mathrm{C1}} = \frac{1}{2}C_1U^2$.

The power delivered by source U is the product of its voltage and the current out of its + terminal.
By KCL at one of the nodes between the current source and voltage source, this current is $-I$.
This gives $P_{\mathrm{U}} = -UI$.

b) $u(0^+) = -U, \quad P_{\mathrm{R3}}(0^+) = \left(\frac{U}{1+\frac{R_1R_2}{(R_1+R_2)R_3}}\right)^2 \Big/ R_3.$

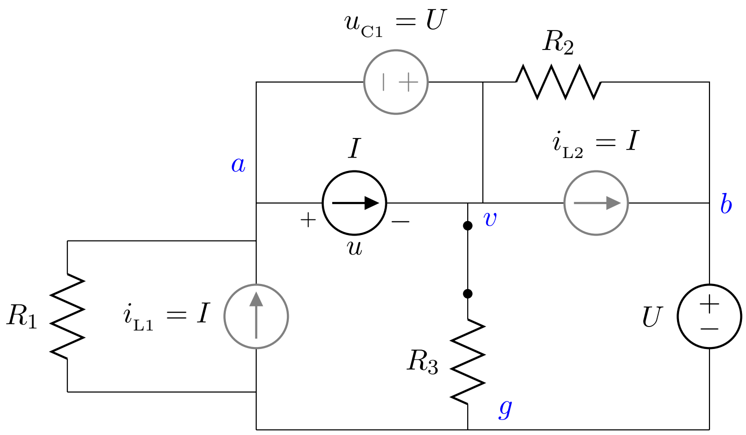
The capacitor voltages and inductor currents have been defined in the above circuit, so that their values can be used in later equilibrium calculations.

By continuity, $u_{\mathrm{C1}}(0^+) = u_{\mathrm{C1}}(0^-) = U$.
The capacitor and current source are in parallel: by KVL, $u(0^+) = -u_{\mathrm{C1}}(0^+) - U$.

Finding the power in R_3 at $t=0^+$ is more difficult. The main step is to find the voltage across, or current through, the resistor R_3. This requires solution of a circuit with several branches.

First, we should draw the circuit in its state at $t=0^+$, replacing the capacitor by a voltage source and the inductors by current sources.

The capacitor voltage was already found, for the solution of $u(0^+)$, above.
The inductor currents are $i_{\mathrm{L1}}(0^+) = i_{\mathrm{L1}}(0^-) = I$ and similarly $i_{\mathrm{L2}}(0^+) = i_{\mathrm{L2}}(0^-) = I$. (This can be deduced from the diagram at $t=0^-$, above. In the equilibrium the resistors in parallel with the inductors do not carry any current, because the inductors in the equilibrium state are like short-circuits and so have no voltage across them. The capacitor is an open circuit, so all the current from source I has to pass through the inductors.)



Nodal analysis. One way to solve the circuit to find the voltage across R_3, is nodal analysis without any further simplifications. The four nodes have been marked with letters. Let's define node g as the reference (ground). Then the voltage across R_3 is v.

Extended nodal analysis would give three KCL equations. If instead we use the concept of supernodes, we get just one equation: nodes a and v become a supernode as they are joined by the voltage source that represents the capacitor; node b is joined to g by the voltage source so it becomes part of the ground supernode, where KCL is not needed.

$$\text{KCL}(a,v)_{\text{out}} \qquad \frac{v-U}{R_1} - I + \frac{v}{R_3} + I + \frac{v-U}{R_2} = 0.$$

Rearranging to find v,

$$v\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right) = U\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

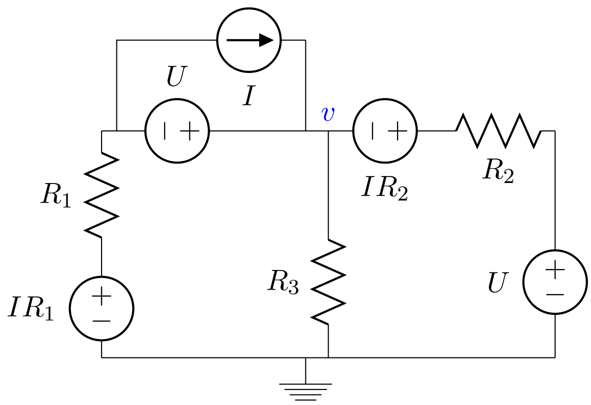
$$v = U\frac{\frac{1}{R_1}+\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}} = U\frac{\frac{R_1+R_2}{R_1R_2}}{\frac{R_1R_2+R_2R_3+R_3R_1}{R_1R_2R_3}} = U\frac{(R_1+R_2)\,R_3}{R_1R_2+(R_1+R_2)\,R_3} = \frac{U}{1+\frac{R_1R_2}{(R_1+R_2)R_3}}.$$

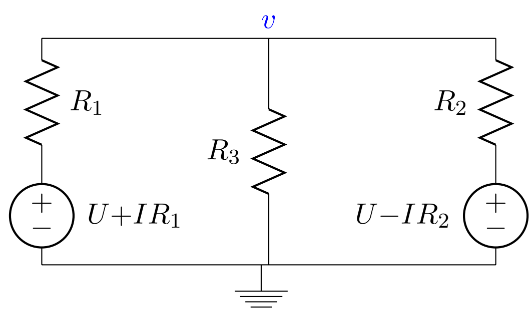
Since v is the voltage across resistor R_3, it is easily used to find $P_{\text{R3}} = v^2/R_3$.

The above is a sufficient solution.
The simplification reached at the right in the above equation isn't required: the first form of the expression for v is acceptable.

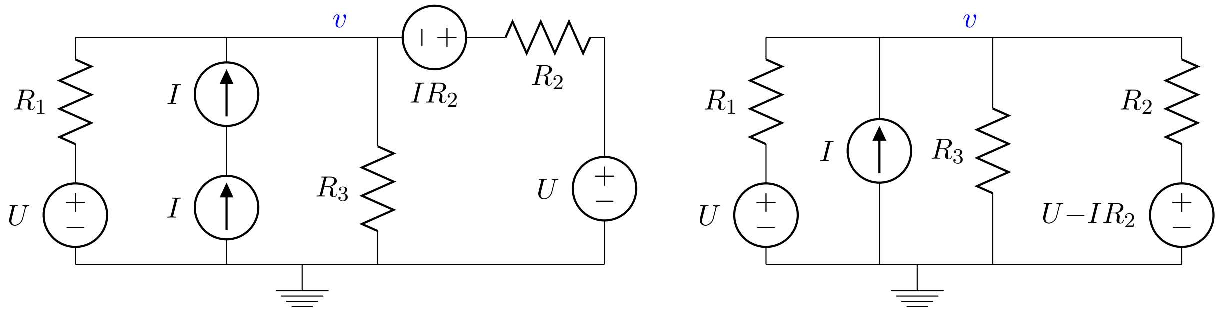
We now look briefly at some other solution methods. In conclusion, however, the supernode nodal analysis used above seems the simplest!

Source transformations. The circuit has two places where a current source and resistor are in parallel. By source-transforming these, we move towards a simpler three-branch circuit. The current source parallel with a voltage source can be ignored (for solution of quantities in the rest of the circuit outside this pair), leaving just three paths for KCL at node v. Try solving it!





Source transformation and simplify current-sources. A quite similar simplification can be done using just one source transformation, if one realises that node 'a' in the original circuit has current sources putting equal currents in *and* out; their total current is therefore zero, so they do not affect the node. These sources can then be separated as shown below, and simplified into one.



(This simplification of current sources is one of many 'theorems' for circuits.
It is an example of the current-source version of Blakesley's source-shift theorem from 1894, which focused on voltage sources, `http://iopscience.iop.org/1478-7814/13/1/307`)

Immediate removal of current sources. We can take it further by noticing in the original circuit that node v also has equal current sources putting current in and out. This 'chain' of three current sources just moves current from node g to node b without actually affecting a or v! Nodes b and g are not affected in their potentials by this current, as they are linked by a voltage source. If we saw this at the start, we could have solved a much simpler circuit, like the second one in the 'Source transformations' method, but with both voltage sources being just U.

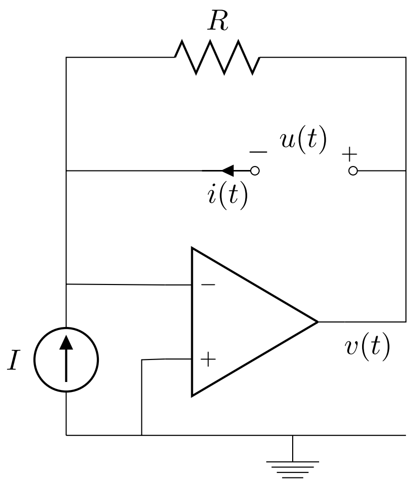
Superposition. Superposition is another approach that could be tried. If the current sources have already been removed (see previous method), then there are only two superposition states to consider.

c) $P_{\mathrm{I}}(\infty) = UI$.

The voltage across the current source is the same when $t \to \infty$ as at $t = 0^-$. This can be seen from KVL around the voltage source, inductors (short-circuit) and current source.

The power coming out of the current source is requested: this is $-uI$.
As $u = -U$ the result is simply UI.

Q5



At times $t \geq 0$, the capacitor is connected across the marked terminals of the circuit shown here on the right. The voltage and current at the terminals are marked as u and i.

The opamp's non-inverting input is at zero potential, so the inverting input is assumed also to be held to zero potential by the feedback: hence $u = v$.

The current through the feedback resistor is then u/R by Ohm's law, but also must be $-I - i$ by KCL, as no current goes into the opamp input. Therefore, $i = -I - u/R$.

Putting these relations together,

$$v(t) = u(t) = -IR - i(t)R \qquad (t > 0).$$

Method 1: ODE solution.
The above expression for $v(t)$ has two unknowns: v and i. In our case, we know that a capacitor is connected. Looking at the directions of v and i (passive convention) we can write the capacitor's equation $i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$ and substitute this for $i(t)$ to get an ODE with $v(t)$ as its dependent variable,

$$v(t) = -IR - RC\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

which in a standard form is

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{RC}v(t) = \frac{-I}{C}$$

with general solution

$$v(t) = -IR + K\mathrm{e}^{-t/RC} \qquad K \text{ to be determined}$$

Before the step-function at $t = 0$ changed the current-source from 0 to I, the equilibrium state was $v(0^-) = 0$, since KCL requires zero current through R if the capacitor is open-circuit (equilibrium) and the current source is zeroed (step-function). The voltage v is the same as the voltage across the capacitor, which is a continuous quantity. Therefore, by continuity $v(0^+) = v(0^-) = 0$. Inserting this into the above solution, at time $t = 0^+$, we find K,

$$v(0^+) = 0 = -IR + K\mathrm{e}^{-0/RC} = -IR + K \quad \Longrightarrow \quad K = IR,$$

from which the function $v(t)$ can be written as

$$v(t) = IR\left(\mathrm{e}^{-t/RC} - 1\right) \qquad (t > 0).$$

Method 2: Equivalent source and function-fit.
Given that this is a first-order circuit, we know that the solution will be of the form

$$v(t) = v(\infty) + \big(v(0^+) - v(\infty)\big)\ \mathrm{e}^{-t/\tau},$$

where $v(\infty)$ is the final value, $v(0^+)$ is the initial value, and τ is the time-constant.

As reasoned in the earlier solution, $v(0^+) = 0$. The final value is an equilibrium, so we assume no current flows in the capacitor; in that case all the current I flows in the resistor R, resulting in $v(\infty) = -IR$. The time-constant can be quite confidently guessed as $\tau = RC$ in this circuit.

However, for more complex circuits (or to justify our guess about the time-constant in this case) it would be sensible to find the Thevenin or Norton equivalent of the circuit that the capacitor is connected to, and then to find the time-constant and final value from this.

The earlier expression $v(t) = -IR - i(t)R$ implies a Thevenin equivalent with $U_{\mathrm{T}} = -IR$ and $R_{\mathrm{T}} = R$.

As expected, this gives a final voltage $v(\infty) = U_{\mathrm{T}} = -IR$ and time-constant $\tau = R_{\mathrm{T}}C = RC$.

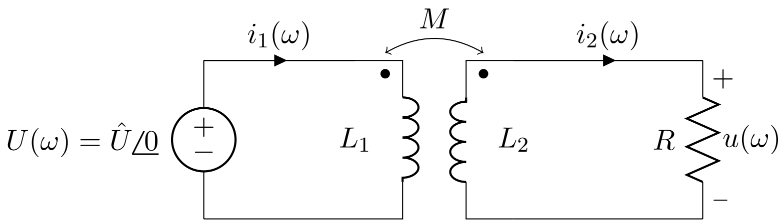
Filling in the initial value, final value and time-constant,

$$v(t) = -IR + (0 - (-IR))\ \mathrm{e}^{-t/RC} = IR\left(\mathrm{e}^{-t/RC} - 1\right) \qquad (t > 0).$$

Q6

We'll do ac analysis (rather than solving by differential equations). Using sine and peak-value, the time-function $\hat{U}\sin\omega t$ is represented by the phasor $U(\omega) = \hat{U}\underline{/0}$.

With currents marked, the circuit is the following:



The equations for mutual inductors, and Ohm's law for the resistor, are:

$$\begin{aligned}
\hat{U}\underline{/0} &= \mathrm{j}\omega L_1 i_1 + \mathrm{j}\omega M(-i_2) \\
u &= \mathrm{j}\omega L_2(-i_2) + \mathrm{j}\omega M i_1 \\
u &= R i_2.
\end{aligned}$$

Substitute the third into the others, to eliminate i_2,

$$\begin{aligned}
\hat{U}\underline{/0} &= \mathrm{j}\omega L_1 i_1 - \mathrm{j}\omega \frac{M}{R} u \\
u &= \mathrm{j}\omega M i_1 - \mathrm{j}\omega \frac{L_2}{R} u
\end{aligned}$$

then the first into the second, to eliminate i_1,

$$u = \mathrm{j}\omega M \left(\frac{\hat{U}\underline{/0}}{\mathrm{j}\omega L_1} + \frac{M}{RL_1} u \right) - \mathrm{j}\omega \frac{L_2}{R} u$$

and rearrange,

$$\frac{M}{L_1}\hat{U}\underline{/0} = \left(1 + \mathrm{j}\omega \frac{L_2}{R} - \mathrm{j}\omega \frac{M^2}{RL_1} \right) u,$$

$$u(\omega) = \frac{\hat{U}\frac{M}{L_1}}{1 + \mathrm{j}\omega\left(\frac{L_2}{R} - \frac{M^2}{L_1 R}\right)} = \frac{\hat{U}\frac{MR}{L_1}}{R + \mathrm{j}\omega\left(L_2 - \frac{M^2}{L_1}\right)}$$

The numerator (top) is purely real. The angle of $u(\omega)$ is therefore the negation of the phase of the denominator,

$$|u(\omega)| = \frac{\hat{U}\frac{M}{L_1}}{\sqrt{1 + \omega^2\left(\frac{L_2}{R} - \frac{M^2}{L_1 R}\right)^2}}, \qquad \underline{/u(\omega)} = -\tan^{-1}\left(\omega\left(\frac{L_2}{R} - \frac{M^2}{L_1 R}\right)\right),$$

Converting back to time, being careful to use the same choice of sine and peak reference,

$$u(t) = \frac{\hat{U}\frac{M}{L_1}}{\sqrt{1 + \omega^2\left(\frac{L_2}{R} - \frac{M^2}{L_1 R}\right)^2}} \sin\left(\omega t - \tan^{-1}\left(\omega\left(\frac{L_2}{R} - \frac{M^2}{L_1 R}\right)\right)\right).$$

Q7

a) In the loop at the left, voltage division gives

$$u_x = \frac{\mathrm{j}\omega L}{R_1 + \mathrm{j}\omega L} u_\mathrm{i}.$$

In the remainder of the circuit, to the right, the voltage u_o is across the parallel impedance of the components C and R_2, with current Gu_x, so

$$u_\mathrm{o} = \frac{R_2 \frac{1}{\mathrm{j}\omega C}}{R_2 + \frac{1}{\mathrm{j}\omega C}} G u_x = \frac{GR_2}{1 + \mathrm{j}\omega C R_2} u_x.$$

Putting these together,

$$H = \frac{u_\mathrm{o}}{u_\mathrm{i}} = \frac{GR_2}{1 + \mathrm{j}\omega C R_2} \cdot \frac{\mathrm{j}\omega L}{R_1 + \mathrm{j}\omega L}$$

b) By a small rearrangement, the above answer

$$H(\omega) = \frac{GR_2}{1 + \mathrm{j}\omega C R_2} \cdot \frac{\mathrm{j}\omega L}{R_1 + \mathrm{j}\omega L} = \frac{\mathrm{j}\omega L G R_2}{(1 + \mathrm{j}\omega C R_2)(R_1 + \mathrm{j}\omega L)} = \frac{GR_2 \; \mathrm{j}\omega L / R_1}{(1 + \mathrm{j}\omega L/R_1)(1 + \mathrm{j}\omega C R_2)},$$

which matches the requested expression if

$$\omega_a = \frac{R_1}{L}, \qquad \omega_b = \frac{1}{CR_2}, \qquad k = GR_2.$$

c) The plot below shows the smooth function of network-function magnitude, and its asymptotic approximation (classic Bode magnitude plot).

The significant details to note are:

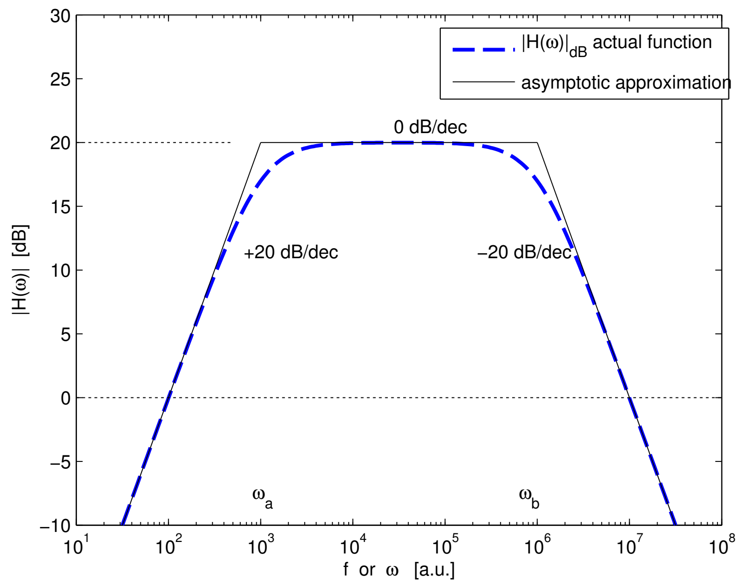
i) the flat part at $\omega_a < \omega < \omega_b$,

ii) this flat part has a level of 20 dB,

iii) slope of 20 dB/decade at $\omega < \omega_a$,

iv) slope of -20 dB/decade at $\omega > \omega_b$.

These details need to be made clear in the submitted solution.



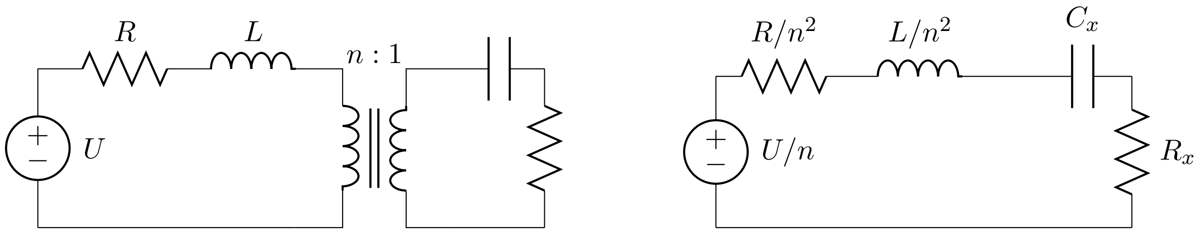
The Bode magnitude diagram can be worked out from the function itself, considering approximations when the frequency is at, or is much above or below, the special frequencies ω_a and ω_b.

Another way one can work out the diagram is by separate treatment of each of the 4 terms in the equation, (k), $(\mathrm{j}\omega/\omega_a)$, $1/(1+\mathrm{j}\omega/\omega_a)$, $1/(1+\mathrm{j}\omega/\omega_b)$, then adding their Bode plots (multiplication of the terms corresponds to addition in the logarithmic scale of the Bode amplitude plot).

Q8

This is a classic "maximum power and transformer" question, based on the ac maximum power, which happens with complex-conjugate matching of the load and source impedances. Different boundaries between 'source' and 'load' can be chosen: for example, the left of the transformer, or the right, or between R and L, or between the unknown components. It is easier to apply the maximum power condition if we remove the transformer and scale (by n) the component values that were on one side of the transformer so that they have the same effect on the components that were on the other side.

'Standard' solution. Before trying a short-cut, we'll go the long way. Let's demonstrate this for the case where we remove the transformer but scale the values of U, R and L by the transformer's ratio, so that they will behave the same when 'seen' by the other components (the unknown components). We can call the unknown components C_x and R_x, and regard them as the load.



By the ac maximum power theorem, we need

$$\left(\frac{R}{n^2} + \mathrm{j}\omega\frac{L}{n^2}\right) = \left(R_x + \mathrm{j}\frac{1}{\mathrm{j}\omega C_x}\right)^* = \left(R_x + \mathrm{j}\frac{1}{\omega C_x}\right),$$

from which (equating real and imaginary parts separately) $R_x = \frac{R}{n^2}$ and $C_x = \frac{n^2}{\omega^2 L}$.
The current in the loop is then

$$i = \frac{U/n}{R/n^2 + R_x + \mathrm{j}\omega L/n^2 + \frac{1}{\mathrm{j}\omega C_x}} = \frac{U/n}{2R/n^2} = \frac{nU}{2R}.$$

The power in the load resistor R_x is

$$P = i^2 R_x = i^2 R/n^2 = \left(\frac{nU}{2R}\right)^2 \frac{R}{n^2} = \frac{U^2}{4R}.$$

Short solution. In this question we do *not have to determine the values of the unknown components*, but only to determine the power into the unknown resistor *given that* the components are chosen to give maximum power. There is a short-cut available.

If we define the source/load break at the left of the transformer, the source is a Thevenin voltage U and impedance $Z = R + \mathrm{j}\omega L$. The load is the combination of the transformer and the two unknown components. It will behave as a plain impedance, and the only part of it that can have non-zero active power is the resistor: the ideal transformer consumes or generates no power (not active or reactive), and the capacitor only generates reactive power.

From this, we see that the maximum possible power from the source will be the power in the unknown resistor in the case when the unknown components are chosen to maximise the power. We don't actually have to analyse the load. The source will deliver maximum power if its load is the complex conjugate of the source impedance: i.e. if the load has a capacitive part that 'cancels' the inductive reactance , and a resistive part equal to R.

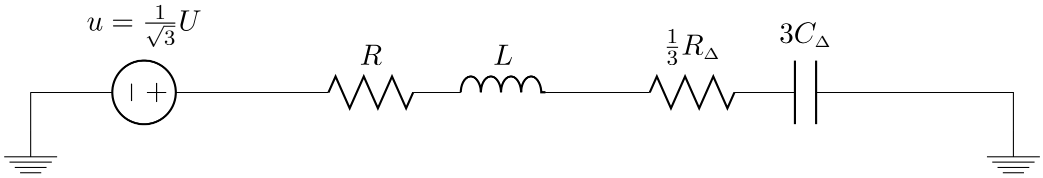
In this case the circuit simplifies to a source U and two series resistors R. Assuming (as usual) that the magnitude of U is in an rms scale, the power in the load resistor is

$$P_{\max} = \left(\frac{U}{2R}\right)^2 R = \frac{U^2}{4R}.$$

Q9

Note that all parts of this question are about a balanced system, in contrast to some recent exams where unbalanced conditions were studied in the last parts of the question.

a) To determine the total active power supplied by the three-phase source, there is the classic difficulty that the delta load and the line impedances form a 'messy' network without obvious parallel connections to simplify. It becomes much easier if the delta is converted to an equivalent star, of components $R_\Delta/3$ and $C_\Delta/3$. Then we can analyse just one phase of this, knowing by symmetry that the other two phases will have similar voltages and currents but with 120° shifts.



In the single-phase circuit above, the source provides a complex power of

$$S_1 = \frac{|u|^2}{Z_{\text{total}}^*} = \frac{\left(\frac{1}{\sqrt{3}}U\right)^2}{R + {}^{R_\Delta}\!/_3 - \mathrm{j}\left(\omega L - \frac{1}{3\omega C_\Delta}\right)}$$

Multiplying by three (to get the total power from the three-phase source in the original question), and taking just the real (active) part,

$$P = \Re\left\{3S_1\right\} = \Re\left\{\frac{U^2}{R + \frac{1}{3}R_\Delta - \mathrm{j}\left(\omega L - \frac{1}{3\omega C_\Delta}\right)}\right\} = \frac{U^2\left(R + \frac{1}{3}R_\Delta\right)}{\left(R + \frac{1}{3}R_\Delta\right)^2 + \left(\omega L - \frac{1}{3\omega C_\Delta}\right)^2}$$

b) In order for the source to supply no reactive power, the total load (the line R,L and the load R_Δ,C_Δ) should appear as a real impedance.

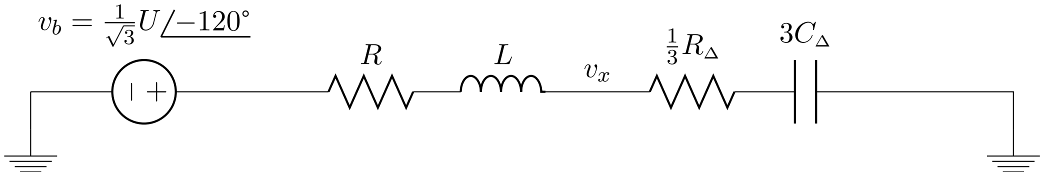
In the single-phase equivalent circuit shown above, this means that the impedances of L and $C_\Delta/3$ should cancel (series resonsance), so that the source 'sees' a pure resistor of $R + R_\Delta/3$. This requires

$$\mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega 3C} = 0$$

and so

$$C = \frac{1}{3\omega^2 L}$$

c) We have to determine a potential, as a phasor. The system is still all balanced, so we can start by analysing a single phase equivalent, similar to what is shown in the solution of subquestion 'a'.



Voltage division gives

$$v_x = v_b \frac{\frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}}{R + \mathrm{j}\omega L + \frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}}$$

Putting in the value of v_b, and trying to neaten it a bit,

$$v_x \;=\; \frac{U}{\sqrt{3}}\angle{-120^\circ} \cdot \frac{\frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}}{R + \mathrm{j}\omega L + \frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}} \;=\; \frac{U}{\sqrt{3}}\angle{-\tfrac{2\pi}{3}} \cdot \frac{R_\Delta - \mathrm{j}\frac{1}{\omega C_\Delta}}{3R + \mathrm{j}\omega 3L + R_\Delta + \frac{1}{\mathrm{j}\omega C_\Delta}}$$

The solution above is sufficient.
We weren't asked for a separate magnitude and angle (polar form) of the result, and we're not going to find a much more compact way of expressing it.

Possible further steps.

If we *were* wanting to get a solution in polar form, one way would be to take the magnitude and angle of each of the three complex terms in the above expression (the voltage which is already in polar form, and the upper and lower impedances in the voltage-division formula) and then join these together:

$$v_x \;=\; \frac{U}{\sqrt{3}} \cdot \sqrt{\frac{R_\Delta^2 + \frac{1}{\omega^2 C_\Delta^2}}{(3R + R_\Delta)^2 + \left(3\omega L - \frac{1}{\omega C_\Delta}\right)^2}}\angle{-\tfrac{2\pi}{3} - \tan^{-1}\tfrac{1}{\omega R_\Delta C_\Delta} - \tan^{-1}\tfrac{3\omega L - \frac{1}{\omega C_\Delta}}{3R + R_\Delta}}.$$

Other apparent simplifications, consisting of several rectangular complex numbers, are

$$v_x \;=\; \frac{U}{\sqrt{3}}\angle{-120^\circ} \cdot \frac{\frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}}{R + \mathrm{j}\omega L + \frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}} \;=\; \frac{-\left(\frac{1}{2\sqrt{3}} + \mathrm{j}\frac{1}{2}\right)U}{1 + \frac{R + \mathrm{j}\omega L}{\frac{R_\Delta}{3} + \frac{1}{\mathrm{j}\omega 3C_\Delta}}} \;=\; \frac{-\left(1 + \mathrm{j}\sqrt{3}\right)\frac{U}{2\sqrt{3}}}{1 + \frac{3R + \mathrm{j}\omega 3L}{R_\Delta + \frac{1}{\mathrm{j}\omega C_\Delta}}}.$$

These do not help to finding the overall magnitude and angle. To do this via a rectangular form, we can try to separate the whole expression into a real and imaginary part. Start by expressing each term in rectangular form, then making the bottom part real by multiplying top and bottom by its complex conjugate.

$$v_x \;=\; \frac{-\frac{U}{2\sqrt{3}}\left(1 + \mathrm{j}\sqrt{3}\right)\left(R_\Delta - \mathrm{j}\frac{1}{\omega C_\Delta}\right)}{3R + \mathrm{j}\omega 3L + R_\Delta + \frac{1}{\mathrm{j}\omega C_\Delta}} \;=\; \frac{U}{2\sqrt{3}} \cdot \frac{-\left(1 + \mathrm{j}\sqrt{3}\right)\left(R_\Delta - \mathrm{j}\frac{1}{\omega C_\Delta}\right)\left(3R + R_\Delta - \mathrm{j}\left(3\omega L - \frac{1}{\omega C_\Delta}\right)\right)}{(3R + R_\Delta)^2 + \left(3\omega L - \frac{1}{\omega C_\Delta}\right)^2}.$$

Expanding out the three complex terms on the top, and grouping them into real and imaginary parts, a rectangular form is found in which the real and imaginary parts are separated.

$$v_x = \frac{-U}{2\sqrt{3}} \cdot \frac{\left(\left(R_\Delta + \frac{\sqrt{3}}{\omega C_\Delta}\right)(3R + R_\Delta) + \left(\sqrt{3}R_\Delta - \frac{1}{\omega C_\Delta}\right)\left(3\omega L - \frac{1}{\omega C_\Delta}\right)\right) + \mathrm{j}\left((3R + R_\Delta)\left(\sqrt{3}R_\Delta - \frac{1}{\omega C_\Delta}\right) - \left(R_\Delta + \frac{\sqrt{3}}{\omega C_\Delta}\right)\left(3\omega L - \frac{1}{\omega C_\Delta}\right)\right)}{(3R + R_\Delta)^2 + \left(3\omega L - \frac{1}{\omega C_\Delta}\right)^2}.$$

From this, the polar form could be expressed. Only one arctangent would be needed to find the angle: that could make it sound simpler than the earlier way in which the angle was found. However, this result is so long that it is clearly neater to express the polar form in terms of two arctangents (as done earlier) instead of going via this expansion into a rectangular complex number!