KTH EI1120 Elkretsanalys (CENMI) KS 1 — 2018-02-01 kl 08–10

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ...

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas. Var tydlig med diagram och definitioner av variabler.

KS1 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-A i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

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1) [4p]

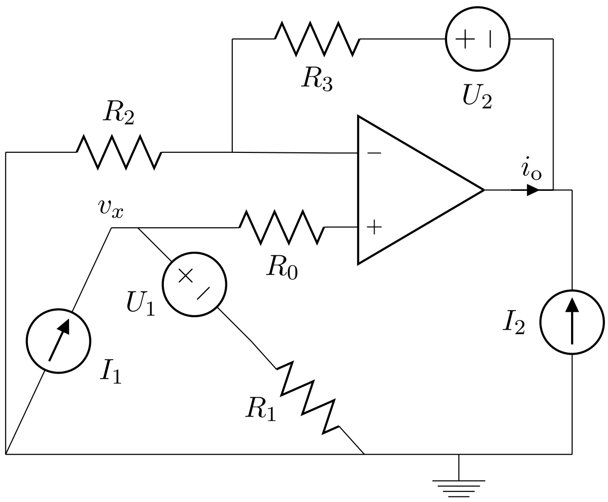
Bestäm följande storheter:

a) [1p] potentialen v_x

b) [1p] effekten absorberad av R_2

c) [1p] strömmen i_o

d) [1p] effekten levererad från källan I_2



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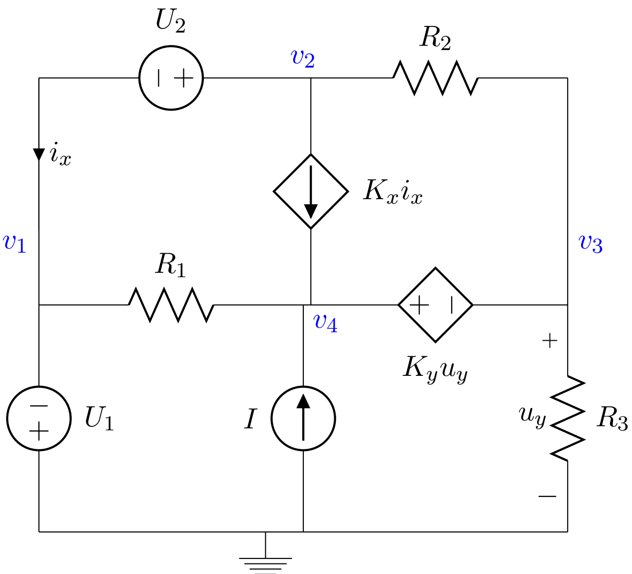
2) [4p+0,5p]

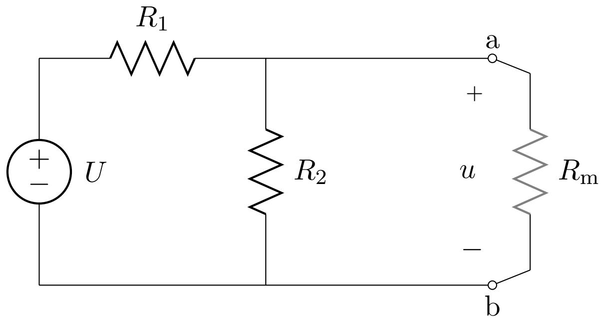
a) [4p] Använd nodanalys för att skriva ekvationer som *skulle kunna lösas* för att få ut de markerade nodpotentialerna v_1, v_2, v_3 och v_4. Du *måste inte* lösa eller förenkla ekvationerna: du behöver bara visa att du kan översätta från kretsen till ekvationerna.

b) [0,5p] Härled v_4. Förslag: supernodmetoden. Obs att det är svårt, för väldigt lite 'extra poäng'. Försök bara om du är klar med allting annan och vill utmanas.



3) [4p]

Kretsen till höger modellerar en uppställning från laboration 1. Motståndet R_{m} representerar en ickeidéal spänningsmätare, vilken kopplas för att mäta spänningen över ett motstånd i en spänningsdelare.



a) [2p] Bestäm Theveninekvivalenten av kretsen förutom R_{m}, sett mellan polerna a-b.

b) [1p] Bestäm u när mätaren är kopplad mellan polerna. (Detta värde är spänningen som mätas.)

c) [1p] Om $R_1 = R_2 = R_{\mathrm{m}}$, bestäm kvoten u/U.
(Alla motstånd kan elimineras från lösningen.)

Short translations of the questions to English:

1. Find:
a) potential v_x
b) the power absorbed by R_2
c) current i_{o}
d) the power delivered from source I_2

2.
a) Write equations that could be solved to find node potentials v_1, v_2, v_3 and v_4. You *are not required to solve them* in this part.
b) For a *small* 'extra point' find potential v_4; this is difficult and is not worth doing unless you have finished everything else and want a challenge. Hint: nodal analysis is probably a good idea, using the supernode method.

3. The diagram models a circuit used in Lab-1. Resistor R_{m} represents a nonideal voltmeter, which is connected so as to measure the voltage across a resistor in a voltage divider.
a) Determine the Thevenin equivalent of the circuit without R_{m} (i.e. without the meter connected) seen at the terminals a-b.
b) What voltage u will be measured when the meter (R_{m}) is connected between these terminals.
c) If $R_1 = R_2 = R_{\mathrm{m}}$, what is the ratio u/U? (All the resistances can be eliminated from this solution.)

Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1120 KS 1 VT18, 2018-02-01)

Q1.

a) $v_x = U_1 + I_1 R_1$
No current flows in the opamp input, so KCL at v_x requires a current of I_1 downwards through R_1. The other side of R_1 is at zero potential, so by KVL the voltages across R_1 and U_1 add to give v_x. (We are careful to check the direction of the current through R_1, i.e. to check that the term for voltage across R_1 is $+I_1R_1$ rather than $-I_1R_1$.)

b) $P_{\mathrm{R2}} = \dfrac{(U_1 + I_1R_1)^2}{R_2}$
R_2 is connected from the opamp's inverting input to zero, so the voltage across it is the same as the potential at the inverting input: we'll call that v_-. The power dissipated in it is therefore v_-^2/R_2.
The opamp is ideal and with negative feedback, so we assume that its inputs have equal potential; the feedback holds the inverting input to follow the non-inverting input. As there is no current in the input, there is no current in R_0, and therefore by Ohm's law no voltage across it. Consequently the potential of the non-inverting input is $v_+ = v_x$. Putting these points together, $v_- = v_+ = v_x$, so the sought power is v_x^2/R_2. Then use the value of v_x from part 'a)'.

c) $i_{\mathrm{o}} = \dfrac{U_1 + I_1R_1}{R_2} - I_2$
In order to use KCL at the opamp's output we need to find the current through U_2. We could do this by calculating the potential at the opamp's output (using KCL at the inverting input), and then finding the current through R_3 by KVL ('potentialvandring') between the opamp output and inverting input. But in this circuit we can solve more directly by noticing that the current to the left in R_2 is v_x/R_2, and that by KCL this must also be the current that is going up from the opamp output into U_2.
Thus, with KCL at the opamp output, $i_{\mathrm{o}} = -I_2 + v_x/R_2$; then substitute for v_x.

d) $P_{\mathrm{I2}} = (U_1 + I_1R_1)\left(1 + \frac{R_3}{R_2}\right) I_2 - U_2 I_2$
This time we can't usefully avoid finding the potential at the opamp's output: let's call it v_{o}.
KCL at the inverting input is $\frac{v_x}{R_2} + \frac{v_x - U_2 - v_{\mathrm{o}}}{R_3} = 0$.
So $v_{\mathrm{o}} = v_x\left(1 + \frac{R_3}{R_2}\right) - U_2$.
As the current source I_2 connects between this potential and zero, v_{o} is the voltage across the current source. For the marked direction of the current, the power delivered by this current source is the product $v_{\mathrm{o}} I_2$.

Q2.

Two examples will be shown. Many variations are possible. The first example is the one that we suggest is probably easiest to do for this type of question, based on systematic use of simple rules. The second is more easily used to obtain the solution requested in part 'b)'.

a) *Write* equations that *could* be solved for all the node potentials.

Extended nodal analysis ("the simple way")

We'll define the unknown currents in the two voltage-sources to be going into the source's + terminal. We'll call them i_α in the independent source U_1, i_x in the independent source U_2 (i_x is already defined so we can use this instead of defining a new same for the same current), and i_β in the dependent source $K_y u_y$.

First we write KCL at all nodes except ground:

$$\begin{aligned}
\mathrm{KCL}(1)_{(\mathrm{out})}: \quad 0 &= -i_\alpha + \frac{v_1 - v_4}{R_1} - i_x && (1)\\
\mathrm{KCL}(2)_{(\mathrm{out})}: \quad 0 &= i_x + K_x i_x + \frac{v_2 - v_3}{R_2} && (2)\\
\mathrm{KCL}(3)_{(\mathrm{out})}: \quad 0 &= -i_\beta + \frac{v_3}{R_3} + \frac{v_3 - v_2}{R_2} && (3)\\
\mathrm{KCL}(4)_{(\mathrm{out})}: \quad 0 &= -I + \frac{v_4 - v_1}{R_1} + i_\beta - K_x i_x && (4)
\end{aligned}$$

These are only 4 equations so far, but with 7 unknowns: v_1, v_2, v_3, v_4, i_α, i_β, i_x.

We can add the further information given by the voltage sources, which provides equations to balance the extra unknowns of the voltage sources' currents,

$$\begin{aligned}
0 - v_1 &= U_1 && (5)\\
v_2 - v_1 &= U_2 && (6)\\
v_4 - v_3 &= K_y u_y && (7)
\end{aligned}$$

One of those equations introduced a further unknown, u_y, which reminds us that we need to define the marked (but unknown) quantities controlling any dependent sources in the circuit:

$$u_y = v_3 \qquad (8)$$

Notice that if we had followed the completely "non-thinking" way of defining a current in source U_2 (e.g. i_γ) then we would also have needed to write an equation that defines what i_x is; that would most simply have been done as $i_x = i_\gamma$. Instead we chose to use the already-marked i_x in our KCL equations, so this quantity is now already defined in the KCL at nodes 1 and 2, and we don't need a further equation to define it (if we tried, it would just be repeating information that already exists in the above equations).

Now there are 8 equations and 8 unknowns.

End. That's it. The above 8 equations are a valid solution to Q2a.

Nodal analysis by simplifications including supernodes

Now we'll try a more solution-friendly approach, even though we aren't required to solve it.

We have a group of three nodes joined by voltage sources: ground, v_1, v_2. For this 'ground supernode' we can write marked the potentials directly:

$$\begin{aligned}
v_1 &= -U_1 && (1)\\
v_2 &= U_2 - U_1 && (2)
\end{aligned}$$

Then we have two more nodes joined by a voltage source (dependent source $K_y u_y$), that form another supernode: v_3, v_4. For this supernode, we let one of the unknown potentials remain: let's choose v_3. Then the potentials of other nodes in that supernode are expressed in terms of that one unknown: in our case, $v_4 = v_3 + K_y u_y$. We want to avoid bringing further unknowns into our equations, so we prefer to express u_y in terms of known quantities or unknowns that are already used in our equations. In this case we see from the diagram that $u_y = v_3$, so we can write:

$$v_4 = v_3 \left(1 + K_y\right). \qquad (3)$$

The preceding three equations define all node potentials except v_3, in terms of known values (components) and the one unknown potential v_3. That suggests we only need to find one further equation to solve, to find v_3. Remembering the rule "KCL at every supernode and separate node except ground", the only KCL we should write in this circuit is at the non-ground supernode, i.e. nodes 3 and 4. KCL is not needed at the ground supernode as the potentials are already fixed.

$$\text{KCL(3,4)}_{\text{(out)}}: \quad 0 \;=\; \frac{v_3}{R_3} - I + \frac{v_3\,(1+K_y)}{R_1} - K_x i_x + \frac{v_3+U_1-U_2}{R_2}$$

But this equation contains the further unknown i_x. Because this is the current in a voltage source, it cannot just be written based on this branch alone, as it could if in a current source or a resistor (in terms of node potentials and resistance). We have to use KCL to determine this current in terms of other branches that the voltage source connects to. Node 1 has a further voltage source (with unknown current), so we'd have to look even further to v_4 and v_3 to find i_x by that approach. Let's look at node 2 instead: KCL(2) is $\frac{v_2-v_3}{R_2} + i_x + K_x i_x = 0$, so

$$i_x = \frac{v_3 - v_2}{(1+K_x)\,R_2} = \frac{v_3+U_1-U_2}{(1+K_x)\,R_2}.$$

Substitute this into the KCL(3,4) equation,

$$\text{KCL(3,4)}_{\text{(out)}}: \quad 0 \;=\; \frac{v_3}{R_3} - I + \frac{v_3\,(1+K_y)}{R_1} - \frac{K_x\,(v_3+U_1-U_2)}{(1+K_x)\,R_2} + \frac{v_3+U_1-U_2}{R_2} \tag{4}$$

and simplify,

$$0 \;=\; \frac{v_3}{R_3} - I + \frac{v_3\,(1+K_y)}{R_1} + \frac{v_3+U_1-U_2}{(1+K_x)\,R_2}$$

End. This set of four equations is a sufficient answer for solving "all the node potentials".

b) Now we are asked to solve (find an expression for) v_4. From the solution in part 'a)' using supernodes, equation (4) can be solved for v_3, and equation (3) lets us find v_4 once v_3 is known.

First we'll rearrange (4) to give v_3,

$$\left(\frac{1+K_y}{R_1} + \frac{1}{(1+K_x)\,R_2} + \frac{1}{R_3}\right) v_3 \;=\; I + \frac{U_2-U_1}{(1+K_x)\,R_2}$$

$$v_3 \;=\; \frac{I + \frac{U_2-U_1}{(1+K_x)R_2}}{\frac{1+K_y}{R_1} + \frac{1}{(1+K_x)R_2} + \frac{1}{R_3}} \;=\; \frac{(1+K_x)\,R_2 I + U_2 - U_1}{1 + \frac{(1+K_y)(1+K_x)R_2}{R_1} + \frac{(1+K_x)R_2}{R_3}}$$

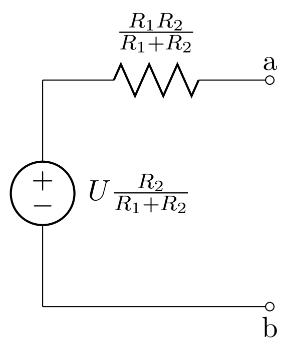
(It's not clear which term above is the nicer way of expressing v_3.)

Substituting with (3),

$$v_4 = (1+K_y)\;v_3 \;=\; \frac{(1+K_y)\left(I + \frac{U_2-U_1}{(1+K_x)R_2}\right)}{\frac{1+K_y}{R_1} + \frac{1}{(1+K_x)R_2} + \frac{1}{R_3}} \tag{5}$$

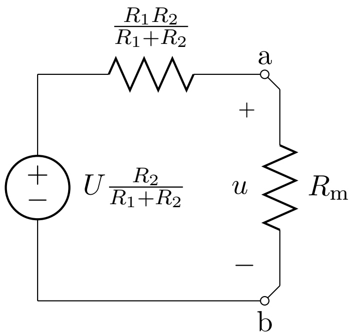
Q3.

a) The Thevenin equivalent of R_1, R_2 and U, between the terminals a-b, is shown on the right.



b) When the voltmeter with resistance R_m is connected between the terminals of the Thevenin equivalent from part 'a)', a voltage divider is formed:

$$u = U\frac{R_2}{R_1+R_2}\frac{R_\mathrm{m}}{R_\mathrm{m}+\frac{R_1R_2}{R_1+R_2}} = \frac{UR_2R_\mathrm{m}}{R_1R_2+R_1R_\mathrm{m}+R_2R_\mathrm{m}}.$$



c) Given that $R_1 = R_2 = R_\mathrm{m}$, let us replace all the resistances in the solution of part 'b)' with 'R', and simplify:

$$u = \frac{URR}{RR+RR+RR} = \frac{U}{3}.$$

As should be familiar from the Lab-1 task, this is the voltage that was seen when the meter and the two resistors in divider to which the meter was coupled were all $10\,\mathrm{M}\Omega$.