KTH EI1120 Elkretsanalys (CENMI) KS 2 — 2018-02-12 kl 08–10

Hjälpmedel: Upp till två A4-ark (båda sidor kan användas) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, osv. Dessa måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas. Var tydlig med diagram och definitioner av variabler.

KS2 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-B i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

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1) [5p]

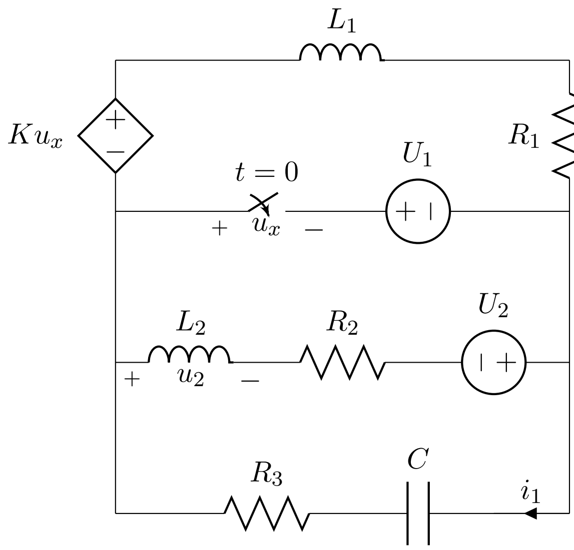
Bestäm följande:

Determine the following:

a) [2p] $u_x(0^-)$

b) [2p] $i_1(0^+)$

c) [1p] $u_2(0^+)$



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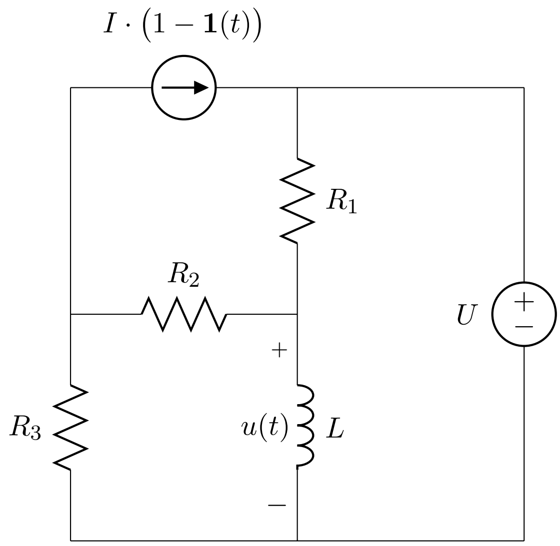
2) [5p]

Bestäm spänningen $u(t)$, för $t > 0$.
($\mathbf{1}(t)$ är enhetsstegfunktionen.)

Find the voltage $u(t)$ for $t > 0$.
($\mathbf{1}(t)$ denotes a unit-step function.)



Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

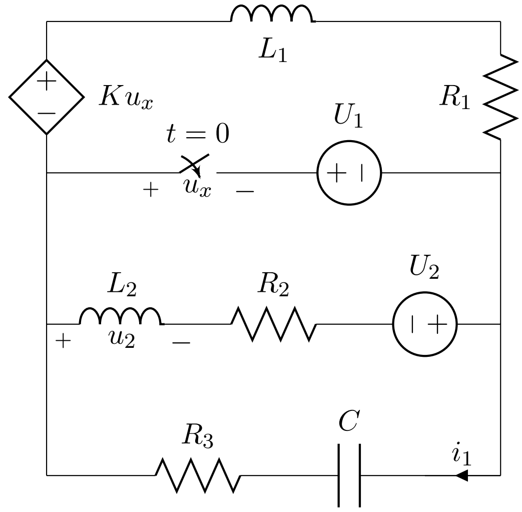
Solutions (EI1120 KS 2 VT18, 2018-02-12)

Q1.

a) $u_x(0^-)$.

The original circuit is shown on the right.

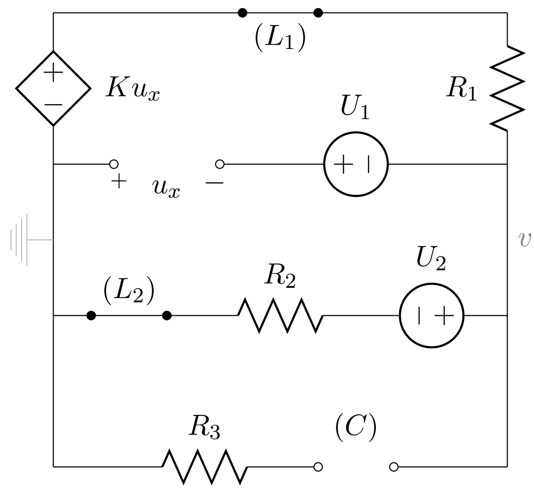
In equilibrium it can be redrawn (right) with the switch in its open state and the capacitors and inductors replaced with their steady-state behaviours of respectively no current and no voltage.



Here is the redrawn circuit for the equilibrium state at $t = 0^-$, when the switch is still open.

The marked i_1 and u_2 are seen to be zero; we've omitted them from the diagram to keep it neat.

We want to find u_x in this state. One choice – often a good one – is a nodal analysis with simplifications. If we mark a ground and a potential v (shown as pale additions in the diagram) we can write KCL at v by looking at the 4 branches between these nodes; two of those have zero current so they don't affect the KCL equation.



$$\text{KCL}(v): \ \frac{v - Ku_x}{R_1} + 0 + \frac{v - U_2}{R_2} + 0 = 0$$

This contains the sought unknown, u_x, but also the extra unknown v that we introduced. We look to the way that u_x is defined in the circuit (this should be relevant to the solution, and the KCL we have written does not describe the definition of u_x across the open circuit).

$$u_x + U_1 = -v \quad\implies\quad v = -U_1 - u_x$$

Substituting this relation into KCL(v) to eliminate v,

$$\frac{-U_1 - u_x - Ku_x}{R_1} + \frac{-U_1 - u_x - U_2}{R_2} = 0$$

and rearranging,

$$u_x\left(\frac{1+K}{R_1} + \frac{1}{R_2}\right) = -\frac{U_1}{R_1} - \frac{U_1 + U_2}{R_2}$$

$$u_x = \frac{-\frac{U_1}{R_1} - \frac{U_1+U_2}{R_2}}{\frac{1+K}{R_1} + \frac{1}{R_2}} = -\frac{\frac{U_1(R_1+R_2)+U_2R_1}{R_1R_2}}{\frac{R_1+(1+K)R_2}{R_1R_2}} = -\frac{U_1(R_1+R_2)+U_2R_1}{R_1+(1+K)R_2}.$$

Alternative method:

Instead of starting with KCL, we could try KVL. The circuit shown for $t = 0^-$ has only one loop that can carry current: we can define that unknown current as i going clockwise. Other branches in this circuit are open circuits, so their current is zero.

$$\text{KVL: } K\,u_x + 0 - iR_1 - U_2 - iR_2 + 0 = 0 \qquad \Longrightarrow \qquad i = \frac{K\,u_x - U_2}{R_1 + R_2}$$

That has two unknowns. We can relate i and u_x through another equation, in the branch with the switch and U_1. This is another KVL.

$$\text{KVL: } 0 = -u_x - U_1 - U_2 - iR_2.$$

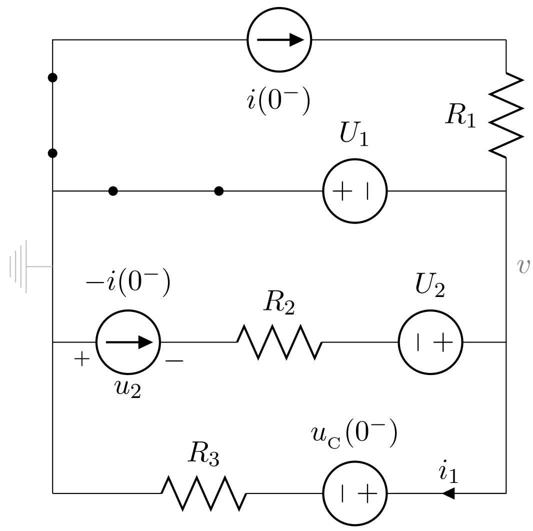
Substituting from the earlier KVL, to eliminate i,

$$0 = -u_x - U_1 - U_2 - \frac{(K\,u_x - U_2)\,R_2}{R_1 + R_2},$$

which is rearranged to

$$u_x \;=\; -\,\frac{U_1 + U_2 - U_2\frac{R_2}{R_1+R_2}}{1 + \frac{KR_2}{R_1+R_2}} \;=\; -\,\frac{U_1\,(R_1+R_2) + U_2R_1}{R_1 + (1+K)\,R_2}.$$

b) $i_1(0^+)$.



At $t = 0^+$ (imediately after the switch closes) the inductors' currents and the capacitor's voltage are the same as before. The diagram on the right shows the circuit at this time.

The switch is closed (short), so there is no voltage across it: $u_x = 0$. This also means that the dependent voltage source $K\,u_x$ becomes fixed to zero, so it can be represented as a short.

The current i that we defined in part 'a)' was flowing around the loop of both inductors. Due to continuity of inductor currents, we can represent the inductors at $t = 0^+$ by current-sources having the value of $i(0^-)$.

The capacitor has a voltage u_C which by continuity is the same as the before the switch closed: $u_C(0^+) = u_C(0^-)$.

Let's start by solving for $i_1(0^+)$ using these symbols i and u_C. Then we can fill in those values later.

The voltage between the left and right nodes is now fixed by the source U_1. (This makes the top branch irrelevant to i_1 and u_2.) KVL around U_1, u_C and R_3 allows us to find the voltage across R_3; then Ohm's law finds the current through it, and KCL says this is the same as the current in the capacitor. Be careful about the signs!

$$i_1(0^+) = \frac{-U_1 - u_C}{R_3}.$$

But now we have to fill in the value of u_C. This is a continuous quantity (voltage across a capacitor). Looking back to the equilibrium state shown in part 'a)', u_C is found by KVL around the lowest loop,

$$u_C = U_2 + iR_2.$$

Into this we substitute $i = \frac{K\,u_x - U_2}{R_1+R_2}$, which we found in the KVL-based method for part 'a)'. Then u_x in the expression for i is substituted with the solution found in part 'a)',

$$i_1(0^+) = \frac{1}{R_3}\left(-U_1 - U_2 - R_2\left(\frac{K\left(-\frac{U_1(R_1+R_2)+U_2R_1}{R_1+(1+K)R_2}\right) - U_2}{R_1+R_2}\right)\right),$$

or more neatly,

$$i_1(0^+) = \frac{-1}{R_3}\left(U_1 + U_2 - R_2\left(\frac{K\left(\frac{U_1(R_1+R_2)+U_2R_1}{R_1+(1+K)R_2}\right) + U_2}{R_1 + R_2}\right)\right).$$

After some 'not inconsiderable' manipulation of the messy result we get

$$i_1(0^+) = -\frac{U_1(R_1 + R_2) + U_2R_1}{(R_1 + (1+K)R_2)\,R_3}.$$

This level of simplification is certainly not required, as it's not entirely obvious at the start whether it's going to simplify so much.

c) $u_2(0^+)$.

Using the same diagram as in part 'b)', KVL in the middle loop gives

$$-U_1 - U_2 + (-i)R_2 + u_2 = 0 \qquad \Longrightarrow \qquad u_2(0^+) = U_1 + U_2 + iR_2.$$

As in part 'b)', we now must use the earlier expressions for 'i' and then for u_x, from part 'a)',

$$u_2(0^+) = U_1 + U_2 + R_2\frac{K\left(-\frac{U_1(R_1+R_2)+U_2R_1}{R_1+(1+K)R_2}\right) - U_2}{R_1 + R_2}$$

This can be simplified to

$$u_2(0^+) = \frac{U_1(R_1 + R_2) + U_2R_1}{R_1 + (1+K)R_2},$$

but again it is not required to go this far with the simplification.

Q2.

Two extreme types of solution will be presented. An intermediate form would be to find a two-pole equivalent (Thevenin or Norton) of everything except the inductor, then to solve the circuit consisting of this equivalent connected to the inductor.

I. Quick 'three-point' method: initial value, final value, time-constant.

We've usually said "solve first for the continuous quantity, then find other quantities from this solution". Following that principle we would first solve for the inductor's current. We'll define this current as $i(t)$, downwards.

The final value is $i(\infty) = \frac{U}{R_1}$.
This is seen from the current source being zero (open circuit) and the other resistors ($R_{2,3}$) being shorted by the inductor, which behaves as a short-circuit as this is an equilibrium state.

The initial value is $i(0^+) = i(0^-) = \frac{U}{R_1} - \frac{IR_3}{R_2+R_3}$.
This is found from equilibrium and continuity. In the equilibrium state at $t = 0^-$, the inductor has zero voltage, i.e. it is like a short-circuit. Its current is by KCL the sum of the currents in R_1 and R_2. The current in R_1 is found directly by Ohm's law, and the current in R_2 is found by current division of I. This is perhaps not obvious from the diagram. Try re-drawing it with the inductor short-circuited. Instead or as well, consider applying nodal analysis with the bottom node as zero and with a supernode around the voltage source: then there is only one other node for which to solve the potential, and that result can be used to find the current in R_2.

The time-constant is $\tau = \frac{L(R_1+R_2+R_3)}{R_1(R_2+R_3)}$.
This is found by the principle that the time-constant is the ratio L/R_{T}, where R_{T} is the equivalent resistance of all the circuit except the inductor, seen from the inductor's terminals. As there are no dependent sources, it is easy to set the independent sources to zero and then simplify the remaining circuit which is simply R_1 in parallel with the series branch of $R_2 + R_3$. This gives $R_{\mathrm{T}} = \frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$.

To make a decaying exponential curve that goes from $i(0^+)$ to $i(\infty)$ with time-constant τ, we take

$$i(t) = i(\infty) + \big(i(0^+) - i(\infty)\big)\mathrm{e}^{-t/\tau}.$$

For our previously found values,

$$i(t) = \frac{U}{R_1} - \frac{IR_3}{R_2+R_3}\exp\left\{-t\,\tfrac{R_1(R_2+R_3)}{L(R_1+R_2+R_3)}\right\} \qquad (t > 0)$$

That is the solution of the continuous quantity (current in the inductor), but we were supposed to find the voltage $u(t)$ across the inductor. These are related by $u = L\frac{\mathrm{d}i}{\mathrm{d}t}$ ... we are of course careful to check that the defined directions of the voltage and current do not require a minus sign in the expression.

$$u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = L \cdot \frac{-IR_3}{R_2+R_3} \cdot \frac{-R_1\,(R_2+R_3)}{L\,(R_1+R_2+R_3)}\exp\left\{-t\,\tfrac{R_1(R_2+R_3)}{L(R_1+R_2+R_3)}\right\}$$

and this simplifies to

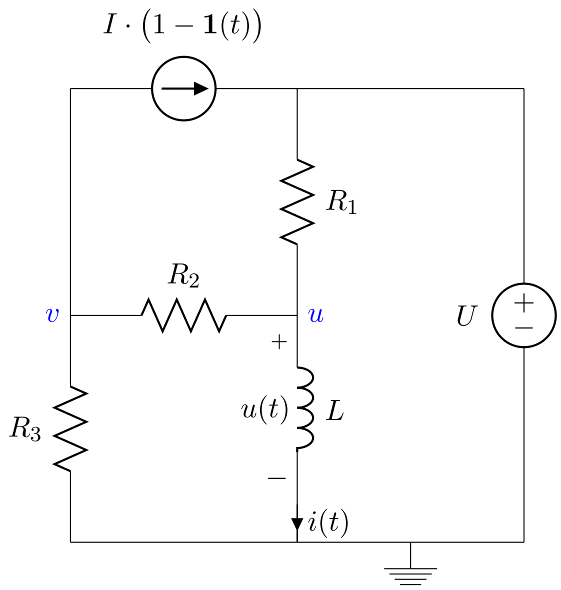
$$u(t) = \frac{IR_1R_3}{R_1+R_2+R_3}\exp\left\{-t\,\tfrac{R_1(R_2+R_3)}{L(R_1+R_2+R_3)}\right\} \qquad (t > 0).$$

II. Forming ODE directly from the circuit.

We'll use nodal analysis as a starting point.

There are four nodes. Two are joined by the voltage source, so they can be treated as a supernode. Let's define the bottom node as the reference (ground); then the node above the voltage source is part of the same supernode: we don't need KCL at either of the nodes in this ground supernode (their potentials are already defined as 0 and U).

Now we write KCL at the two remaining nodes. One has a potential already defined by the marked 'u'. The other we can define as 'v'. We'll define the current downwards through the inductor as i.



$$\mathrm{KCL}(u): \quad 0 \;=\; i + \frac{u-v}{R_2} + \frac{u-U}{R_1}$$
$$\mathrm{KCL}(v): \quad 0 \;=\; \frac{v}{R_3} + \frac{v-u}{R_2} + I \cdot (1 - \mathbf{1}(t))$$

These are two equations in three unknowns. We also can include the inductor's equation,

$$u \;=\; L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

It links u and i and makes our problem solvable, but the derivative term means that a differential equation has to be solved.

Let us first eliminate v from the two KCL equations.
Rearrange KCL(v)

$$0 = \left(\frac{1}{R_2} + \frac{1}{R_3}\right) v - \frac{u}{R_2} + I \cdot (1 - \mathbf{1}(t)),$$

then rearrange KCL(u)

$$v = R_2 \left(u \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + i - \frac{U}{R_1} \right),$$

and substitute for v,

$$0 = \left(\frac{1}{R_2} + \frac{1}{R_3}\right) R_2 \left(u \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + i - \frac{U}{R_1} \right) - \frac{u}{R_2} + I \cdot (1 - \mathbf{1}(t)).$$

Now substitute with $u = L\frac{\mathrm{d}i}{\mathrm{d}t}$, to get a differential equation in the continuous quantity i.
As we are only interested in finding the result for $t > 0$, we can simplify the step-function term, since $(1 - 1(t)) = 0$ for $t > 0$,

$$\left(\frac{R_2 + R_3}{R_2 R_3} R_2 \cdot \frac{R_1 + R_2}{R_1 R_2} - \frac{1}{R_2} \right) L \frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R_2 + R_3}{R_2 R_3} R_2 i(t) = \frac{R_2 + R_3}{R_1 R_2 R_3} U.$$

Now rearrange in a standard form,

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R_1(R_2 + R_3)}{L(R_1 + R_2 + R_3)} i(t) = \frac{(R_2 + R_3)U}{(R_1 + R_2 + R_3)\, L},$$

which has the following general solution, where K is a constant to be determined,

$$i(t) = \frac{U}{R_1} + K \exp\left\{-t\tfrac{R_1(R_2+R_3)}{L(R_1+R_2+R_3)}\right\}.$$

To find K we need the initial condition. Consider the equilibrium at $t = 0^-$. In this case the inductor has zero voltage (it is like a short-circuit). Thus $u(0^-) = 0$. If we take the earlier equation that combined both KCL equations and eliminated v, we can use it at $t = 0^-$ by setting $u = 0$ and $(1 - 1(t)) = 1$, to find the equilibrium current,

$$0 = \left(\frac{1}{R_2} + \frac{1}{R_3}\right) R_2 \left(i(0^-) - \frac{U}{R_1}\right) + I, \qquad \Longrightarrow \qquad i(0^-) = \frac{U}{R_1} - \frac{R_3}{R_2 + R_3} I.$$

Due to the continuity of inductor current, $i(0^+) = i(0^-)$.
Putting this into the ODE solution,

$$i(0^+) = \frac{U}{R_1} - \frac{R_3}{R_2 + R_3} I = \frac{U}{R_1} + K\,\mathrm{e}^0 \qquad \Longrightarrow \qquad K = -\frac{R_3}{R_2 + R_3} I.$$

Inserting this value of K to the general solution of the ODE gives (as expected) the same result as was found by the first method, of initial value, final value, time constant. Then, as was done there, the voltage $u(t)$ can be found from the current $i(t)$.

A comment. This circuit looks simple: there are no dependent sources, and only six components. But it needs surprisingly much algebra to solve it in a general way; one attempted explanation is that the topology initially has no series or parallel branches that can be simplified (after the current source becomes zero, it is simpler, with two series-connected resistors).

III. Thevenin equivalent.

Several approaches can be used here to find the Thevenin or Norton equivalent of the circuit that the inductor 'sees' at its terminals.

Considering $t > 0$, when the current source has zero value, the short-circuit current where the inductor is connected is simply $i_{\mathrm{sc}} = U/R_1$. The open-circuit voltage at the same terminals is found by voltage division, $u_{\mathrm{oc}} = U(R_2 + R_3)/(R_1 + R_2 + R_3)$. The equivalent resistance is then $u_{\mathrm{oc}}/i_{\mathrm{sc}}$, which could alternatively be found by the method of setting sources to zero and simplifying, as used in our first example solution method.

Alternatively, being more equation-oriented than diagram-simplification-oriented, we could start in the same way as the second example solution method (forming ODE directly), to rearrange and combine the two KCLs, which gives a relation between u and i at the terminals where the inductor is connected. This equation can be rearranged in the form $u = U_{\mathrm{T}} - iR_{\mathrm{T}}$ to identify the Thevenin parameters directly, or it can have current and voltage set to zero to identify respectively u_{oc} and i_{sc}.

Having found the Thevenin equivalent, it is still necessary to consider also the circuit *before* the step function, to get the right initial condition for the inductor's current. The Thevenin equivalent found above is not appropriate for this time, since it is based on the current source having value 0. Approaches as shown in methods I or II can be used.