

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

**Studenter från EI1110 svarar bara på Del C, växelström!**

**Hjälpmedel:** Upp till tre A4-ark (båda sidor kan användas) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, osv. Dessa måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex.  $R$  för ett motstånd,  $U$  för en spänningskälla,  $K$  för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas. Var tydlig med diagram och definitioner av variabler.

*Tips:* Dela tiden mellan talen. *Senare deltal brukar vara svårare* att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räknande av betyg: Låt  $A$ ,  $B$  och  $C$  vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s.  $A=12$ ,  $B=10$ ,  $C=18$ . Låt  $a$ ,  $b$  och  $c$  vara poängen man får i dessa respektive delar i tentan, och  $a_k$  vara poängen man fick från kontrollskrivning KS1, och  $b_k$  poängen från KS2, och  $h$  bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a, a_k)}{A} \geq 0,4 \quad \& \quad \frac{\max(b, b_k)}{B} \geq 0,4 \quad \& \quad \frac{c}{C} \geq 0,3 \quad \& \quad \frac{\max(a, a_k) + \max(b, b_k) + c + h}{A + B + C} \geq 0,5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget  $F_x$  registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Observera att nu i VT18 omgången har vi givit möjlighet för studenter på EI1120 att klara av alla poäng från tal 9, om trefas växelström, genom godkänd datoruppgift (hemuppgift 13).

Nathaniel Taylor (073 949 8572)

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## Del A. Likström

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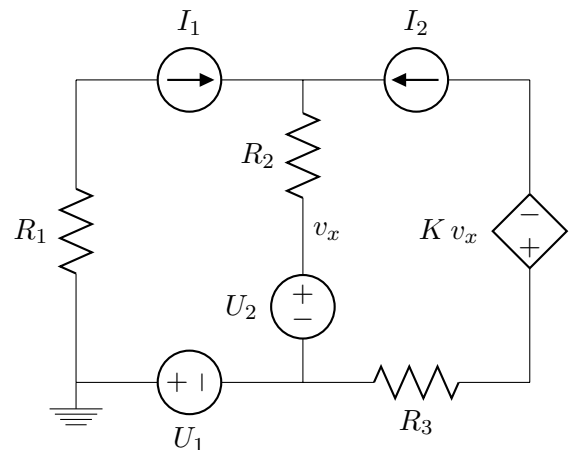
1) [4p] Bestäm effekterna absorberad av följande komponenterna:

a) [1p]  $R_1$ .

b) [1p]  $R_3$ .

c) [1p]  $U_2$ .

d) [1p]  $I_2$ .

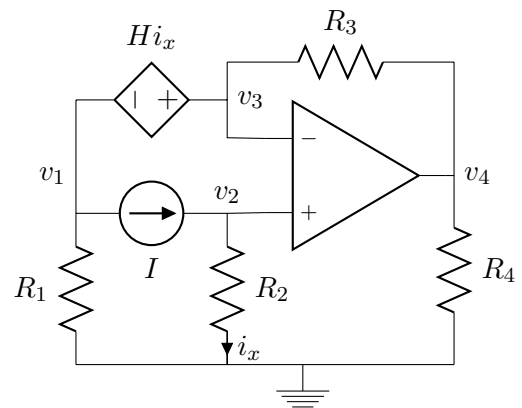


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2) [4p]

a) [3p] Använd nodanalys för att skriva ekvationer som skulle kunna lösas för alla fyra nodpotentialerna  $v_1, v_2, v_3, v_4$ . Du måste inte lösa ekvationerna.

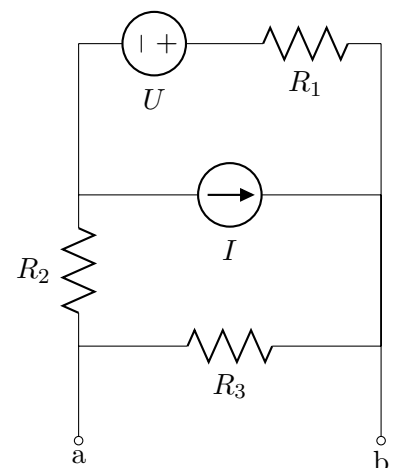
b) [1p] Bestäm  $v_4$  (valfri metod).



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3) [4p]

Bestäm den maximala effekten som kretsen kan leverera från polerna a-b (till någonting som man ansluter till dessa poler).



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## Del B. Transient

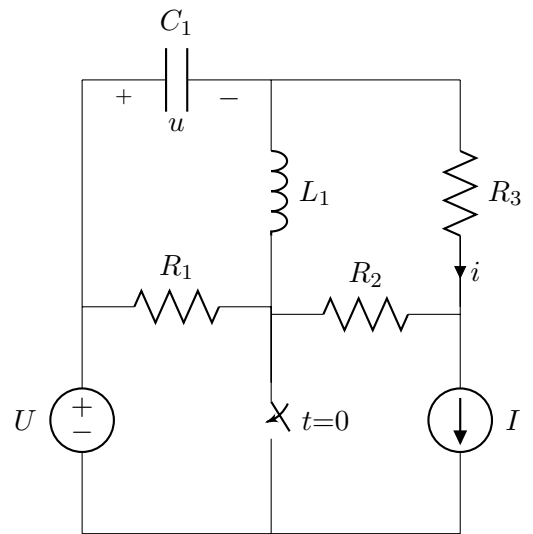
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4) [5p] Bestäm  $u$  och  $i$  vid följande tider:

a) [2p]  $t = 0^-$ .

b) [2p]  $t = 0^+$ .

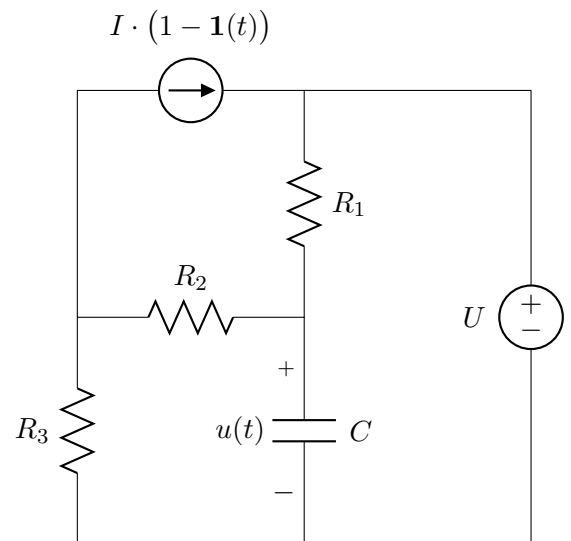
c) [1p]  $t \rightarrow \infty$ .



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5) [5p]

Bestäm  $u(t)$ , för  $t > 0$ .



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## Del C. Växelström

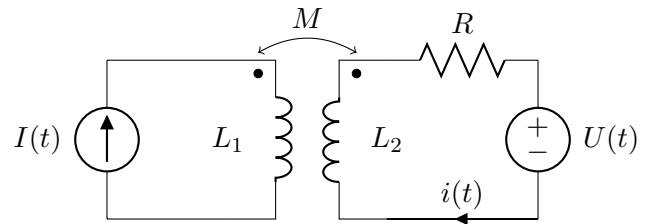
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6) [4p]

Källorna har följande tidsfunktioner:

$$\begin{aligned} I(t) &= \hat{I} \cos(\omega t) \\ U(t) &= \hat{U} \sin(\omega t) \end{aligned}$$

Bestäm  $i(t)$ .



7) [4p]

a) [2p] Bestäm kretsens nätverksfunktion,

$$H(\omega) = \frac{v_x(\omega)}{v_i(\omega)}.$$

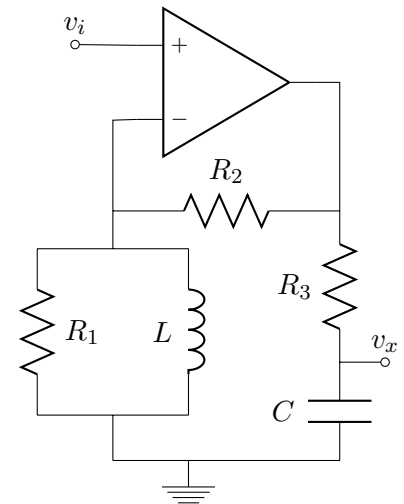
b) [2p] Svaret till delat 'a' kan skrivas på följande formen,

$$H(\omega) = \frac{1 + j\omega/\omega_b}{(j\omega/\omega_a)(1 + j\omega/\omega_c)}.$$

Skiss ett Bode amplituddiagram av funktionen ovan.

Anta att:  $\omega_a \ll \omega_b \ll \omega_c$ .

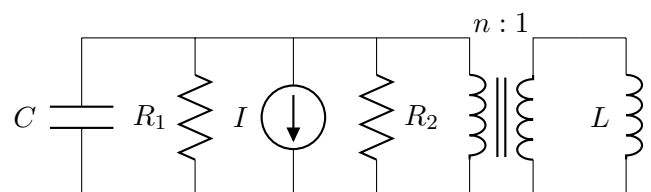
Markera viktiga frekvenser och lutningar.



8) [4p]

Källan har vinkelfrekvens  $\omega$ .

a) [3p] Bestäm  $R_2$  och  $L$  för att maximera den aktiva effekten som levereras till  $R_2$ .



b) [1p] Ersätt spolen  $L$  med en kortslutning, och transformatorn med kopplade spolar  $L_1$  (vänster) och  $L_2$  (höger). Bestäm motståndet  $R_2$  och kopplingskoefficienten  $k$  mellan spolerna, för att maximera den aktiva effekten som levereras till  $R_2$ .

9) [6p]

Spänningskällorna utgör en balanserad trefaskälla med huvudspänning  $U$  och vinkelfrekvens  $\omega$ .

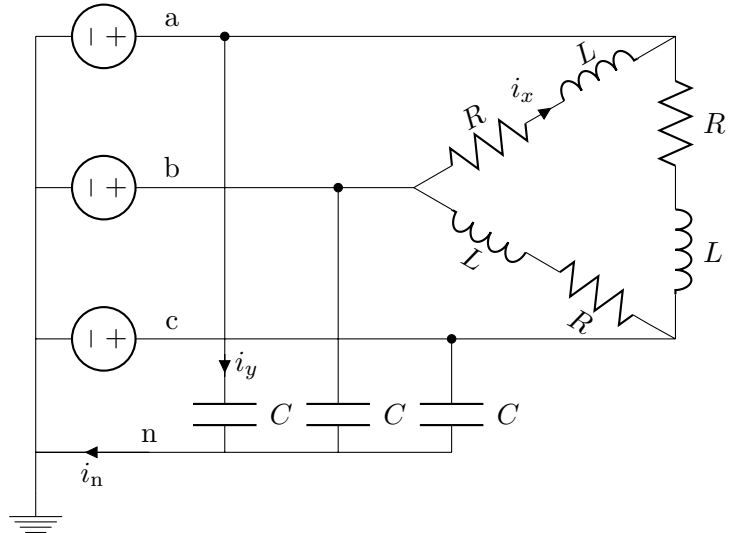
a) [1p] Bestäm strömmen  $i_n$ .

b) [1p] Ge ett exempel på tre fasvektorer (komplexa tal) som skulle kunna beskriva potentialerna i noderna 'a', 'b' och 'c'. Det finns flera möjliga svar.

c) [1p] Bestäm den reaktiva effekten levererad till den delta-anlutna lasten (alla  $R$  och  $L$  komponenterna).

d) [2p] Bestäm effektfaktorn (PF) hos hela lasten som den trefas spänningskällan matar.

e) [1p] Bestäm fasvinkeln hos strömmen  $i_y$  relativ till strömmen  $i_x$ .



Slut ... men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren!

## Solutions (EI1120 TEN1 VT18, 2018-03-13)

### Q1

a)  $P_{R1} = I_1^2 R_1$ .

The current source and resistor are in series; KCL determines the resistor's current.

b)  $P_{R3} = I_2^2 R_3$ .

Similar to the case in 'a)', although not perhaps so obvious, as there is a dependent source between the current source and the resistor.

c)  $P_{U2} = (I_1 + I_2)U_2$ .

The power delivered to this source is the product of its voltage and the current into the terminal that is the positive reference of that voltage (i.e. 'passive convention'). The current downwards in this voltage source is found directly by KCL in the top node.

d)  $P_{I2} = I_2(KU_1 - (1+K)U_2 - I_1R_2 - I_2(R_2+R_3))$ .

The power into this source is the product of its current and the voltage of its right-hand side relative to its left-hand side (as the current is marked right-to-left).

KVL around the right-hand loop gives:  $P_{I2} = I_2(-K(U_2 - U_1) - I_2R_3 - U_2 - (I_1 + I_2)R_2)$ , where the first term comes from the KVL relation  $v_x = -U_1 + U_2$ .

### Q2

a) Two possible methods are shown below, for writing suitable equations.

#### Extended nodal analysis.

Start with KCL at every node except the reference. This circuit has two voltage sources: the dependent source and the opamp output. We can define unknown currents in these: we'll call them  $i_\alpha$  into the +pole of the dependent source, and  $i_o$  out of the opamp.

$$\text{KCL(1): } 0 = \frac{v_1}{R_1} + I - i_\alpha \quad (1)$$

$$\text{KCL(2): } 0 = \frac{v_2}{R_2} - I \quad (2)$$

$$\text{KCL(3): } 0 = i_\alpha + \frac{v_3 - v_4}{R_3} \quad (3)$$

$$\text{KCL(4): } 0 = \frac{v_4 - v_3}{R_3} + \frac{v_4}{R_4} - i_o \quad (4)$$

Now we have 4 equations, in 6 unknowns. Beyond the 4 unknown node-potentials, there are the currents in the voltage sources; this hints to us that we should look to the voltage sources to provide further equations. The dependent source sets a relation between two node-potentials:

$$v_3 - v_1 = Hi_x. \quad (5)$$

However, this new equation introduced a further unknown, the marked quantity  $i_x$ . This needs to be described in terms of the existing defined quantities, in order for the equations to describe all the relevant detail of the circuit. We can express  $i_x$  as

$$i_x = \frac{v_2}{R_2}. \quad (6)$$

The opamp is the other voltage source. In contrast to normal voltage sources, where we can directly express their voltage in terms of a constant or another circuit quantity, the opamp's voltage is whatever value is needed to force the inputs to be equal. The equation that it guarantees for us is therefore equality of the input potentials:

$$v_3 = v_2. \quad (7)$$

The above equations are sufficient.

**Nodal analysis: simplify on the way, e.g. supernode**

Another approach is to try to avoid extra equations. For nodes joined by a voltage source, we write just one KCL for those nodes together, and define one of the potentials, expressing the other[s] in terms of this potential and the source voltage. This is equivalent to summing the separate KCL equations and thereby eliminating the unknown current in a voltage source. We can also try to avoid writing extra equations for controlling variables, by substituting suitable expressions at the start.

$$\text{KCL(1\&3): } 0 = \frac{v_1}{R_1} + I + \frac{v_1 + H \frac{v_2}{R_2} - v_4}{R_3} \quad (1)$$

$$\text{KCL(2): } 0 = \frac{v_2}{R_2} - I \quad (2)$$

$$\text{KCL(4): } 0 = \frac{v_4 - v_3}{R_3} + \frac{v_4}{R_4} - i_o \quad (3)$$

b) One possible approach is to take the equations from answer ‘a’ and solve them for  $v_4$ . In this circuit it may be easier just to reason about the circuit, solving for each potential. Potential  $v_2$  is determined directly by  $v_2 = IR_2$ ; this is seen from KCL, as the opamp input has no current. Then by the assumption of an ideal opamp with negative feedback, we know  $v_3 = v_2 = IR_2$ . The dependent source’s voltage is determined by the relation  $i_x = I$ , so that  $H i_x = HI$ . The current through the dependent source determines  $v_4$ : if this current is  $i_\alpha$  (into the + side) then  $v_4 = v_3 + i_\alpha R_3$ . To find what  $i_\alpha$  is, we must look to the other side of the dependent source, where the potential must be  $v_1 = v_3 - H i_x$  which can be written  $v_1 = IR_2 - IH$ . KCL gives that  $i_\alpha = \frac{v_1}{R_1} = \frac{I(R_2 - H)}{R_1}$ .

Then  $v_4 = IR_2 + \frac{IR_3(R_2 - H)}{R_1}$ .

**Q3**

The maximum power that can be obtained from a two-terminal linear circuit is the product of half the short-circuit current and half the open-circuit voltage, i.e.  $P_{\max} = \frac{u_{oc} i_{sc}}{4}$ . One example of how this can be achieved is when a resistance equal to the circuit’s source resistance (its Thevenin or Norton resistance) is connected. If we find the Thevenin or Norton equivalent of this circuit between terminals a-b, then we can express its maximum possible power output as  $\frac{U_T^2}{4R_T}$ .

The open-circuit voltage and short-circuit current are quite similar to derive for this circuit: in both cases one can solve a three-branch circuit, using for example a KCL equation, or source-transformation or superposition; the difference is whether  $R_2$  and  $R_3$  form one branch, or whether it is only  $R_2$  (when  $R_3$  is shorted). Let’s take the short-circuit case, which should be a little easier.

Write KCL in the top left-hand node, for the case of a short-circuit between a-b. To avoid defining further variables, we can start by expressing the voltage between this node and the node on the right as  $i_{sc} R_2$ , where  $i_{sc}$  is the short-circuit current from a to b.

$$\text{KCL: } i_{sc} + I + \frac{i_{sc} R_2 + U}{R_1} = 0$$

$$i_{sc} = - \frac{U + IR_1}{R_1 + R_2}$$

The open-circuit voltage could be found similarly, using the value  $R_2 + R_3$  instead of just  $R_2$  when writing KCL, then using Ohm’s law to find the voltage across  $R_3$  from the current through the  $R_2$ - $R_3$  branch. This gives

$$u_{oc} = - R_3 \frac{U + IR_1}{R_1 + R_2 + R_3}$$

Alternatively, we can directly find the equivalent resistance (Thevenin resistance) of the circuit between a-b by setting the independent sources to zero: as there are no dependent sources, the resulting circuit contains just resistors, making it easy to solve. With the voltage source represented as a short-circuit, and the current source as an open circuit, the resulting resistance between a-b is  $R_3$  in parallel with the series pair  $R_1 + R_2$ ,

$$R_{\text{eq}} = \frac{(R_1 + R_2)R_3}{R_1 + R_2 + R_3}.$$

As expected, this resistance fulfils the condition  $R_{\text{eq}} = \frac{u_{\text{oc}}}{i_{\text{sc}}}$ .

We only needed to find any two of the above solutions ( $u_{\text{oc}}$ ,  $i_{\text{sc}}$ ,  $R_{\text{eq}}$ ), in order to answer the question.

The final answer is:

$$P_{\text{max(ab)}} = \frac{u_{\text{oc}}}{2} \cdot \frac{i_{\text{sc}}}{2} = \frac{u_{\text{oc}}^2}{4R_{\text{eq}}} = i_{\text{sc}}^2 \frac{R_{\text{eq}}}{4} = \frac{R_3 (U + IR_1)^2}{(R_1 + R_2)(R_1 + R_2 + R_3)}.$$

(In the above explanation the names equivalent resistance (or source resistance)  $R_{\text{eq}}$ , and  $u_{\text{oc}}$  and  $i_{\text{sc}}$  could instead have been called the Thevenin or Norton resistance, Thevenin voltage, and Norton current, respectively.)

#### Q4

a) Original equilibrium, before the switch closes at  $t = 0$ :

$$u(0^-) = IR_1$$

$$i(0^-) = I \frac{R_2}{R_2 + R_3}$$

b) Immediately after the change, with continuity:

$$u(0^+) = IR_1$$

$$i(0^+) = \frac{U + I(R_2 - R_1)}{R_2 + R_3}$$

c) New equilibrium:

$$u(\infty) = U$$

$$i(\infty) = I \frac{R_2}{R_2 + R_3}$$

#### Q5

Before  $t = 0$  the current source is active, with current  $I$ .

At  $t = 0^-$ , the capacitor behaves as an open circuit (equilibrium). Solving the circuit in this condition gives:

$$u(0^-) = \frac{U(R_2 + R_3) - IR_1R_3}{R_1 + R_2 + R_3}.$$

By continuity,  $u(0^+) = u(0^-)$ .

Times  $t > 0$  are after the current source becomes zero. A zeroed current source can alternatively be modelled as an open circuit. Drawing the circuit for this situation, the capacitor is connected to the three resistors and the voltage source. These components, as 'seen' from the capacitor's terminals, have a Thevenin equivalent of

$$U_{\text{T}} = \frac{R_2 + R_3}{R_1 + R_2 + R_3}U \quad \text{and} \quad R_{\text{T}} = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}.$$

The Thevenin resistance allows us to find the time-constant:  $\tau = CR_{\text{T}}$ .

In the final equilibrium, as  $t \rightarrow \infty$ , no further change has happened, so the same Thevenin equivalent is valid. Drawing this connected to the capacitor, which is an open-circuit (equilibrium), the voltage across the capacitor is  $u(\infty) = U_{\text{T}}$ .

Using the method of initial value, final value and time-constant, the time function we're seeking is

$$u(t) = u(\infty) + (u(0^+) - u(\infty)) e^{-t/CR_{\text{T}}} \quad (t > 0).$$



As the final answer, we must put the earlier expressions back, to express this in terms of the given quantities instead of our own definitions:

$$u(t) = \frac{R_2 + R_3}{R_1 + R_2 + R_3} U - \frac{R_1 R_3}{R_1 + R_2 + R_3} I \exp\left(-t \frac{R_1 + R_2 + R_3}{R_1(R_2 + R_3)C}\right) \quad (t > 0).$$

### Q6

Both sources are sinusoidal, with the same frequency: we can use one solution, representing each source as a phasor. (As this is the ac part of the exam, we assume ‘sinusoidal steady-state’ conditions, i.e. transients when ‘turning on’ the circuit have now died away, so just the sinusoidal solution is needed.)

Let’s take  $\cos(\omega t)$  as our reference, so that this function corresponds to a phasor with zero angle. Then the  $\sin(\omega t)$  function corresponds to an angle of  $-\pi/2$ . Then we can express the two sources as phasors at angular frequency  $\omega$ :

$$I(\omega) = \hat{I}, \quad U(\omega) = -j\hat{U}.$$

To get a solution of everything in the circuit, we could write and solve 4 equations: two for KVL in the loops, and two for the mutual inductors. In the specific case we only want the current in the right-hand loop, *and* the current in the left-hand loop is already determined by the current source, so we can work with just 2 of the equations:

$$\begin{array}{ll} \text{KVL}_{(\text{right})} & u_2 = U + iR \\ \text{mutual inductance} & u_2 = j\omega L_2(-i) + j\omega MI \end{array}$$

Putting these together and rearranging,

$$i = \frac{j\omega MI - U}{R + j\omega L_2}$$

into which the values of the phasors can now be inserted,

$$i = \frac{j\omega M\hat{I} + j\hat{U}}{R + j\omega L_2}$$

The magnitude and angle of this  $i(\omega)$  are

$$|i| = \frac{\omega M\hat{I} + \hat{U}}{\sqrt{R^2 + \omega^2 L_2^2}} \quad \text{and} \quad \angle i = \frac{\pi}{2} - \tan^{-1} \frac{\omega L_2}{R}.$$

As we used a cosine reference (and peak-value reference) to translate from time-functions to phasors, the same choice must also be used to find the time-function of  $i(t)$ :

$$i(t) = \frac{\omega M\hat{I} + \hat{U}}{\sqrt{R^2 + \omega^2 L_2^2}} \cos\left(\omega t + \frac{\pi}{2} - \tan^{-1} \frac{\omega L_2}{R}\right).$$

### Q7

a) One possible approach here is nodal analysis on the entire circuit. We will use another approach, by identifying that  $R_3$  and  $C$  form a load that does not affect the rest of the circuit, and that the remaining components form a standard non-inverting amplifier. The reason the load does not affect the rest of the circuit is that it connects to a pure voltage source: it is between the opamp output and the ground node. (If you use a ‘supernode’ method to do a nodal analysis, you will see this from the fact that the output node will be ‘part of the ground supernode’ and will not need a KCL.)

The non-inverting amplifier has a relation  $\frac{v_o}{v_i} = \frac{Z_1 + Z_2}{Z_1}$  where  $Z_1$  is the lower and  $Z_2$  the upper impedance in the voltage divider between the opamp’s output and inverting input. In our circuit,  $Z_1 = \frac{j\omega L R_1}{R_1 + j\omega L}$ , and  $Z_2 = R_2$ , giving the relation

$$\frac{v_o}{v_i} = \frac{R_2 + \frac{j\omega L R_1}{R_1 + j\omega L}}{\frac{j\omega L R_1}{R_1 + j\omega L}} = R_2 \frac{R_1 + j\omega L}{j\omega L R_1} + 1 = \frac{R_1 + j\omega L + j\omega L R_1 / R_2}{j\omega L R_1 / R_2} = \frac{1 + j\omega L / (\frac{R_1 R_2}{R_1 + R_2})}{j\omega L / R_2}$$

The potential  $v_x$  is in the middle of a voltage divider that is connected between ground and  $v_o$ . The relationship of  $v_x$  to  $v_o$  is therefore

$$\frac{v_x}{v_o} = \frac{\frac{1}{j\omega C}}{R_3 + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega C R_3}.$$

Putting the above two results together,

$$\frac{v_x}{v_i} = \frac{v_x}{v_o} \cdot \frac{v_o}{v_i} = \frac{1 + j\omega L / (\frac{R_1 R_2}{R_1 + R_2})}{(j\omega L / R_2) (1 + j\omega C R_3)}.$$

You were not required to rearrange the equations to this classic form. However, doing so might have helped you check that your answer agrees with the clue given in part ‘b’.

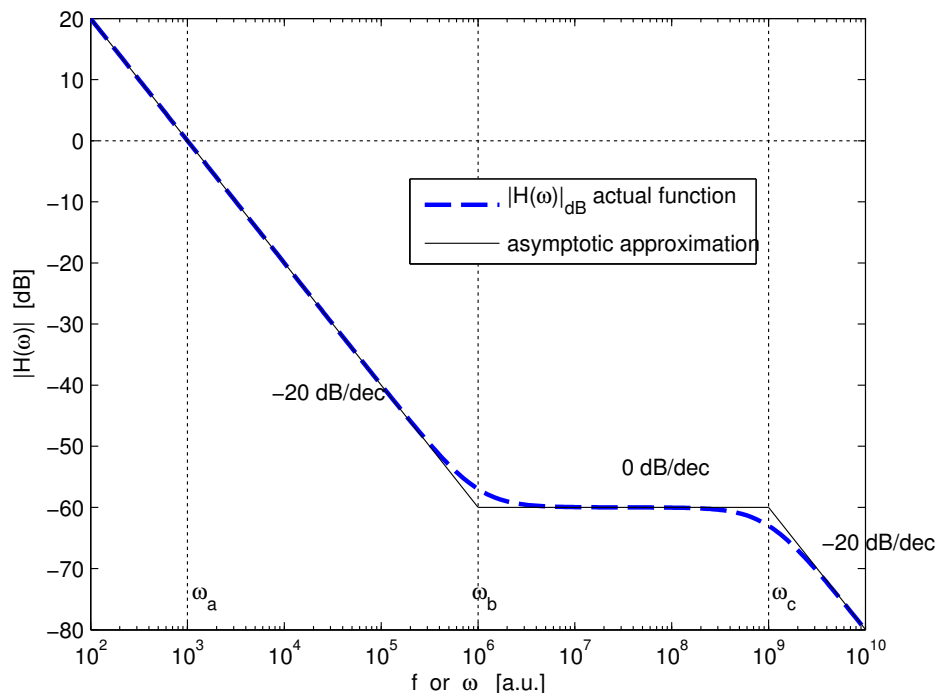
b) There are three terms:

$|j\omega/\omega_a|$  is a single line at  $-20$  dB/decade slope, passing through 0 dB at  $\omega_a$ ,

$|1 + j\omega/\omega_b|$  is flat at 0 dB until  $\omega_b$ , but then slopes up at 20 dB/decade,

$|\frac{1}{1+j\omega/\omega_c}|$  is flat at 0 dB until  $\omega_c$ , but the slopes down.

The following plot shows these features. We have chosen 1000 as the ‘very big difference’ to fulfil  $\omega_a \ll \omega_b \ll \omega_c$ ; the numerical values on the dB axis are based on this assumption. You did not have to put any numbers except 0 dB (at  $\omega_a$ ).



If you’re being very observant, you perhaps noticed that the assumption (anta att  $\omega_a \ll \omega_b \ll \omega_c$ ) contradicts the actual circuit in part ‘a’. From the original equations, we see  $\omega_a = \frac{R_2}{L}$  whereas  $\omega_b = \frac{R_1 R_2}{L(R_1 + R_2)}$ . The parallel combination  $\frac{R_1 R_2}{R_1 + R_2}$  cannot be larger than  $R_2$ , if all resistance values are positive. So in fact we expect  $\omega_b$  to be less than  $\omega_a$ . But never mind . . . that wouldn’t give such a nice plot! Part ‘b’ didn’t explicitly say that the assumption relates to our circuit: it’s just what you are told to assume for the diagram.

## Q8

a) We are asked to choose two of these components (resistor, inductor) to maximise the active power transferred to that same resistor. This sounds like a classic maximum power question, even if its diagram is perhaps not so obvious.

Let's consider the inductor and transformer together as an inductor  $n^2L$ . This inductance, parallel with resistance  $R_2$ , can be seen as a load impedance (or admittance). The remainder of the circuit,  $C$ ,  $R_1$  and  $I$ , would then be seen as the source from which the load is to extract maximum power: in fact, it is basically a Norton source, with an impedance made of two parallel components.

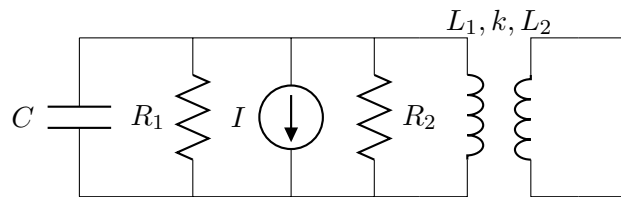
It's easier to use admittances here, since all the components are in parallel. For maximum active power transfer to the load admittance  $Y_l$  from the source with admittance  $Z_s$  we need  $Y_l = Y_s^*$ ,

$$\frac{1}{R_2} + \frac{1}{j\omega n^2L} = \frac{1}{R_1} - j\omega C,$$

which requires

$$R_2 = R_1 \quad \text{and} \quad L = \frac{1}{\omega^2 n^2 C}.$$

b) Now the circuit is changed to the one shown below.



We are given  $L_1$  and  $L_2$ , but the coupling coefficient between these can be chosen. This could for example be that we have two coils and can choose their distance and relative orientation. Recall the relation  $M = k\sqrt{L_1L_2}$  between mutual and self inductances and coupling. Let us see how the whole combination of the two coils and the short circuit behaves, when seen at the terminals of the left coil. Calling this left coil's voltage  $u$ , and writing the short-circuited coil's voltage as 0, the mutual-inductance equations are

$$\begin{aligned} u &= j\omega L_1 i_1 + j\omega M i_2 \\ 0 &= j\omega L_2 i_2 + j\omega M i_1 \end{aligned}$$

from which

$$u = j\omega \left( L_1 - \frac{M^2}{L_2} \right) i_1 = j\omega \left( L_1 - \frac{k^2 L_1 L_2}{L_2} \right) i_1 = j\omega (1 - k^2) L_1.$$

This tells us that at the terminals of  $L_1$ , the behaviour of the circuit made from  $L_1$ ,  $L_2$ , their coupling  $k$  and the shortcircuit on  $L_2$ , is simply the behaviour of an inductor  $(1 - k^2)L_1$ .

Following the same principle as in part 'a', but with  $(1 - k^2)L_1$  instead of  $n^2L$ ,

$$1 - k^2 = \frac{1}{\omega^2 C L_1} \quad \implies \quad k = \sqrt{1 - \frac{1}{\omega^2 C L_1}}.$$

### Q9

a)  $i_n = 0$ .

This marked current is between the neutrals (star-points) of the star-connected source and star-connected load. The source is known to be a balanced three-phase voltage, and the loads are balanced. So no neutral current is expected.

b)  $\left\{ \frac{U}{\sqrt{3}} \angle 0, \frac{U}{\sqrt{3}} \angle -\frac{2\pi}{3}, \frac{U}{\sqrt{3}} \angle \frac{2\pi}{3} \right\}$ . The absolute angle is free to be arbitrarily chosen, as long as the three phasors keep equal relative angles. The phase-rotation is not specified either, so it doesn't matter which of the phasors is said to be for phase a, b or c of the source. The line voltage magnitude is  $U$ , so the phase-voltage magnitude of the star-connected source should be  $U/\sqrt{3}$ .

c) Each phase of the delta load has impedance  $R + j\omega L$ , and is connected to the line-voltage,  $U$ . For three of these phases, the complex power is

$$S_\Delta = 3 \frac{U^2}{R - j\omega L}.$$

The reactive power delivered to this load is the imaginary part,

$$Q_{\Delta} = \Im S_{\Delta} = \frac{3U^2\omega L}{R^2 + \omega^2 L^2}.$$

d) The power factor of the whole load (the delta and star parts) could be found by converting the loads to an equivalent impedance, and taking the ratio of the impedance's real part to its magnitude. To find an equivalent impedance, we'd probably choose to look at one phase of a star-connected load, for which we'd convert the delta to a star also.

Alternatively, we can look straight at the powers. The complex power into the delta load was already found as a step in the solution of part 'c'. The complex power into the star load is  $S = 3 \frac{(U/\sqrt{3})^2}{\left(\frac{1}{j\omega C}\right)^*} = -j\omega C U^2$ . (The '-j' shows that it is a pure reactive power, and is coming out from the capacitors.) Summing these two loads,

$$S = 3 \frac{U^2}{R - j\omega L} - j\omega C U^2$$

The power factor is  $\frac{P}{|S|}$ , which with  $S$  defined as above means:

$$\text{PF} = \frac{\Re S}{|S|} = \frac{\frac{3U^2 R}{R^2 + \omega^2 L^2}}{\sqrt{\left(\frac{3U^2 R}{R^2 + \omega^2 L^2}\right)^2 + \left(\frac{j\omega L}{R^2 + \omega^2 L^2}\right)^2}}$$

e) Absolute choice of angle isn't important as we're asked for the relative phase. We can define any angle we like, e.g. to set the voltage of source 'a' to zero phase, as long as the relative angles are a balanced three-phase set.

In the course we've had a default assumption of phase-rotation 1,2,3 or a,b,c. The rotation *does* matter for this question. If the opposite is assumed (a,c,b), then in this question the angle will be negated. You would get full points either way, as long as it's clear what you assumed.

Let's define the source voltages as  $\{v_a, v_b, v_c\} = \frac{U}{\sqrt{3}} \angle \{0, \frac{-2\pi}{3}, \frac{-4\pi}{3}\}$ .

Then,

$$i_x = \frac{\frac{U}{\sqrt{3}} \angle \frac{-2\pi}{3} - \frac{U}{\sqrt{3}} \angle 0}{R + j\omega L} \quad \text{and} \quad i_y = \frac{\frac{U}{\sqrt{3}} \angle 0}{\frac{1}{j\omega C}}.$$

We simplify these into pure polar form, by looking at a drawing of the phasors, or playing with some sin and cos functions. (For example, draw a diagram with  $v_a$  along the real axis, and thus  $v_b$  at  $-120^\circ$ , then draw in the line-voltage  $v_b - v_a$  by joining these two points: it will have a phase of  $-150^\circ$  if you look at it as a phasor going from  $v_a$  to  $v_b$ , which is what  $v_b - v_a$  means.)

$$i_x = \frac{U \angle \frac{-5\pi}{6}}{R + j\omega L} = \frac{U}{\sqrt{R^2 + \omega^2 L^2}} \angle \frac{-5\pi}{6} - \tan^{-1} \frac{\omega L}{R} \quad \text{and} \quad i_y = \frac{U\omega C}{\sqrt{3}} \angle \frac{\pi}{2}$$

The requested angle is then,

$$\angle i_y - \angle i_x = \frac{\pi}{2} - \frac{-5\pi}{6} + \tan^{-1} \frac{\omega L}{R} = \frac{8\pi}{6} + \tan^{-1} \frac{\omega L}{R} = \frac{-4\pi}{6} + \tan^{-1} \frac{\omega L}{R}.$$

The radian angles such as  $\frac{-4\pi}{6}$  could equally well have been expressed as  $-120^\circ$ ; we don't care which is used.