

Tentan har 9 tal i 3 delar: tre tal i del A (12p), två i del B (10p) och fyra i del C (18p).

Hjälpmedel: Upp till tre A4-ark (båda sidor kan användas) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, osv. Dessa måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. *Senare deltal brukar vara svårare* att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Räkande av betyg: Låt A , B och C vara de maximala möjliga poängen från delarna A, B och C i tentan, d.v.s. $A=12$, $B=10$, $C=18$. Låt a , b och c vara poängen man får i dessa respektive delar i tentan, och a_k vara poängen man fick från kontrollskrivning KS1, och b_k poängen från KS2, och h bonuspoängen från hemuppgifterna. Godkänd tentamen (och därigenom hel kurs) kräver:

$$\frac{\max(a, a_k)}{A} \geq 0,4 \quad \& \quad \frac{\max(b, b_k)}{B} \geq 0,4 \quad \& \quad \frac{c}{C} \geq 0,3 \quad \& \quad \frac{\max(a, a_k) + \max(b, b_k) + c + h}{A + B + C} \geq 0,5.$$

Betyget räknas också från summan över alla delar och bonuspoäng, d.v.s. sista termen ovan, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Observera att nu i VT18 omgången har vi givit möjlighet för studenter på EI1120 att klara av alla poäng från tal 9, om trefas växelström, genom godkänd datoruppgift (hemuppgift 13).

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Del A. Likström

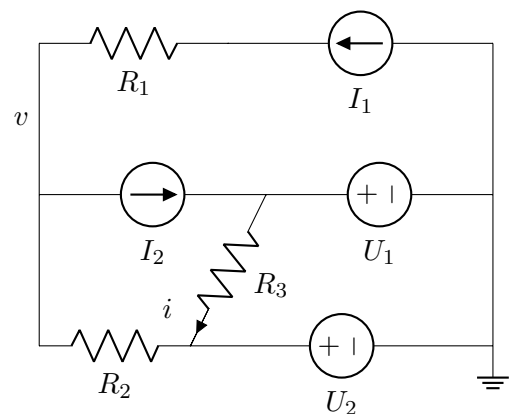
1) [4p] Bestäm följande:

a) [1p] Effekten absorberad av R_2 .

b) [1p] Strömmen i .

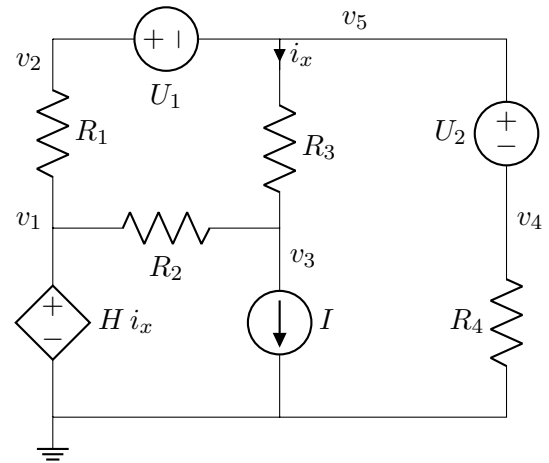
c) [1p] Potentialen v .

d) [1p] Effekten levererad av källan U_1 .



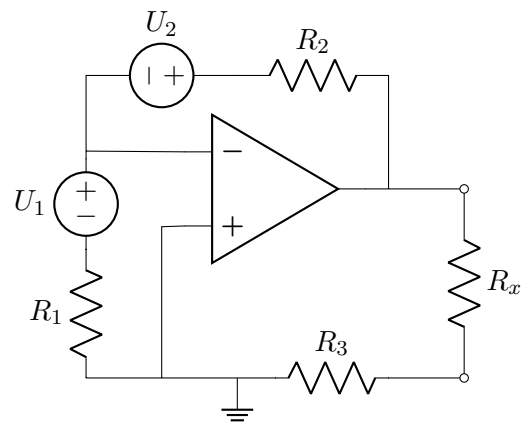
2) [4p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för alla fem nodpotentialerna v_1 , v_2 , v_3 , v_4 , v_5 . Du måste inte lösa ekvationerna.



3) [4p]

Motståndet R_x är valt för att det ska få så mycket effekt som möjligt från resten av kretsen. Hur mycket effekt får det?

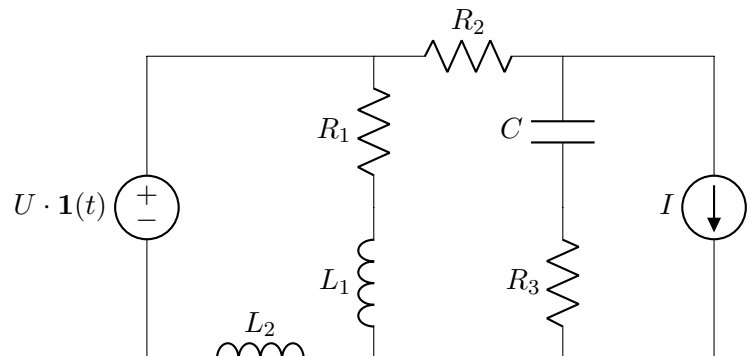


Del B. Transient

4) [5p] Bestäm följande:

a) [3p] vid $t = 0^+$,
effekten absorberad av R_1 , R_3 , L_2 :
 $P_{R_1}(0^+)$, $P_{R_3}(0^+)$, $P_{L_2}(0^+)$.

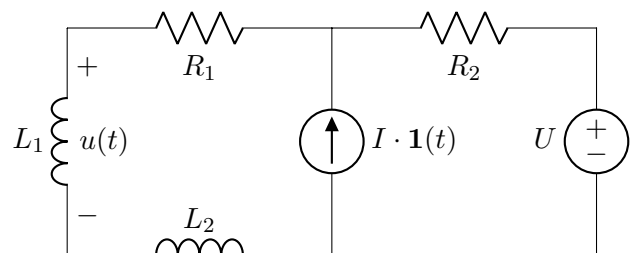
b) [2p] $t \rightarrow \infty$,
energin lagrad i L_2 , C :
 $W_{L_2}(\infty)$, $W_C(\infty)$.



5) [5p]

Bestäm $u(t)$, för $t > 0$.

Obs. att spolarna är seriekopplade och därför kan behandlas som en ekvivalent under lösningen, förutom sista steget där man beräknar spänningen över en enda spole.



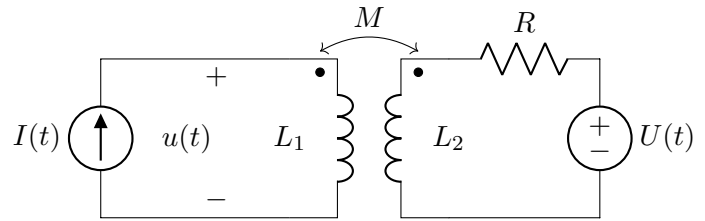
Del C. Växelström

6) [4p]

Källorna har följande tidsfunktioner:

$$I(t) = \hat{I} \cos(\omega t)$$

$$U(t) = \hat{U} \sin(\omega t)$$



Bestäm $u(t)$.

7) [4p]

a) [3p] Rita ett Bode amplituddiagram av funktionen $H(\omega) = \frac{(1 + j\omega/\omega_2)(1 + j\omega/\omega_3)}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_3)}$.

Anta att: $\omega_1 \ll \omega_2 \ll \omega_3$.

Markera viktiga frekvenser och lutningar.

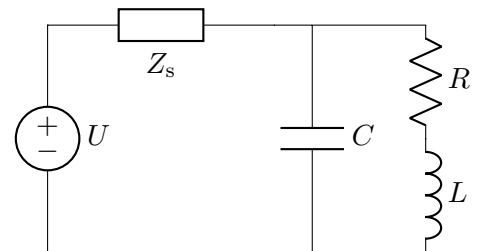
b) [1p] Rita en krets som har nätverksfunktionen $\frac{1}{1 + j\omega/\omega_0}$.

Visa tydligt vilka storheter i kretsen som är 'in' och 'ut' för nätverksfunktionen.

Visa också hur ω_0 kan uttryckas som funktion av komponentvärden i kretsen.

8) [4p]

Källan har vinkelfrekvens ω . Den är kopplad genom en ledning med impedans Z_s , till en last R och L med tillhörande kondensator C .



a) [2p] Bestäm C så att effektfaktorn för hela lasten av C , L , R tillsammans blir 1.

b) [2p] Visa hur man skulle kunna bestämma värden av R och C som maximerar den aktiveffekt som levereras till lasten. Du får lämna ekvationer oförenklade, d.v.s. med R och C på flera ställen.

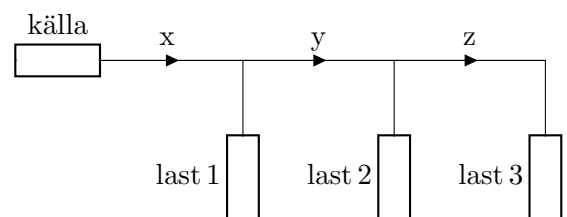
9) [6p] Kretsen visar en trefas källa, och tre trefas laster (9 impedanser totalt). De är alla balanserade. Linjerna mellan dessa representerar trefas anslutningar bestående av tre fasledare med försumbara impedanser.

Källa: huvudspänning U , vinkelfrekvens ω .

Last 1: skenbareffekt $|S_1|$, effektfaktor PF_1 (lagging).

Last 2: aktiveffekt P_2 , effektfaktor PF_2 (lagging).

Last 3: skenbareffekt $|S_3|$, fas θ_3 av ström relativ till spänning.



a) [2p] Vilken reaktiveffekt flöder vid markeringen 'y'?

b) [2p] Vilken strömamplitud (absolutbelopp) finns i varje ledare i linjen vid markeringen 'x'?

c) [2p] Låt oss kalla de tre fasledarna i linjen för a,b,c, och definera vinklarna så att spänningen av 'a' relativ till 'b' är $U/30^\circ$. Fasföljden är a,b,c. Vad är strömmens fasvinkel i fasledare 'b' vid markeringen 'z'?

Slut ... men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren!

Solutions (EI1120 TEN1 VT18, 2018-05-29)

Q1

a) $P_{R_2} = (I_1 - I_2)^2 R_2.$

KVL at the left node. Order of $I_1 - I_2$ or $I_2 - I_1$ doesn't matter, as it's squared.

b) $i = \frac{U_1 - U_2}{R_3}.$

KVL in loop at bottom right. The order (signs) of U_1 and U_2 *do* matter.

c) $v = U_2 + (I_1 - I_2) R_2.$

Potential-change from earth node to node v ; avoid current sources as we don't directly know their voltages, which should *not* be assumed to be zero (a surprisingly common mistake).

d) $P_{U_1} = \frac{U_1(U_1 - U_2)}{R_3} - U_1 I_2.$

The power out from source U_1 is the product of its voltage and the current defined out of its '+' terminal. This current is found by KCL in the central node of the circuit: $i - I_2$. Substitute the expression for i , and multiply by U_1 .

Q2

Two possible methods are shown below, for writing suitable equations.

Extended nodal analysis.

Start with KCL at every node except the reference. This circuit has three voltage sources: two independent and one dependent. We can define unknown currents in these: let's define them into the +-pole of the source, and call them i_α , i_β and i_γ in respectively sources U_1 , U_2 and $H i_x$.

$$\text{KCL(1): } 0 = i_\gamma + \frac{v_1 - v_2}{R_1} + \frac{v_1 - v_3}{R_2} \quad (1)$$

$$\text{KCL(2): } 0 = \frac{v_2 - v_1}{R_1} + i_\alpha \quad (2)$$

$$\text{KCL(3): } 0 = I + \frac{v_3 - v_1}{R_2} + \frac{v_3 - v_5}{R_3} \quad (3)$$

$$\text{KCL(4): } 0 = \frac{v_4}{R_4} - i_\beta \quad (4)$$

$$\text{KCL(5): } 0 = \frac{v_5 - v_3}{R_3} - i_\alpha + i_\beta. \quad (5)$$

Now we have 5 equations, in 8 unknowns. The 5 unknown node-potentials and 5 KCL equations would give a well defined solution. But we the 3 voltage sources have added unknowns; this hints that we should look to the voltage sources to provide corresponding further equations. The source set the following relation between pairs of node-potentials:

$$v_2 - v_5 = U_1 \quad (6)$$

$$v_5 - v_4 = U_2 \quad (7)$$

$$v_1 = H i_x. \quad (8)$$

However, this last of these introduced a *further* unknown, the marked quantity i_x . This needs to be described in terms of the existing defined quantities, in order for the equations to describe all the relevant detail of the circuit:

$$i_x = \frac{v_5 - v_3}{R_3}. \quad (9)$$

The above equations are a sufficient solution.

Nodal analysis: simplify on the way, e.g. supernode

Another approach is to try to reduce the number of equations from the start, instead of ending up as in the above example, with lots of simpler equations to solve.

If we follow this principle, and use the idea of supernodes, then we end up with just two equations to solve, then after solving them we find other potentials by simple relations given by the voltage sources.

The nodes v_2 , v_4 and v_5 are joined by voltage sources, so we use just one of these potentials as an unknown, and write the others in terms of that by adding or subtracting the voltage source values, as in equations (6) and (7) above. Let's keep v_4 , since the direction of the voltage sources is such that we can then express v_5 and v_2 without negative signs:

$$v_5 = v_4 + U_2 \quad (1)$$

$$v_2 = v_4 + U_1 + U_2. \quad (2)$$

The node v_1 is joined to the zero node by the dependent voltage source, so we can immediately write $H i_x$ instead of v_1 . However, we prefer to avoid the further variable i_x : we saw that it can be written instead as $\frac{v_5 - v_3}{R_3}$, and since we've decided to use only the potential v_4 in the top supernode, we use (1) to substitute for v_5 ,

$$v_1 = H \frac{v_4 + U_2 - v_3}{R_3}. \quad (3)$$

Now we can write KCL for the non-reference supernodes and other nodes: we have just one such supernode and one other node.

$$\text{KCL}(2\&4\&5): 0 = \frac{v_4 + U_1 + U_2 - H \frac{v_4 + U_2 - v_3}{R_3}}{R_1} + \frac{v_4 + U_2 - v_3}{R_3} + \frac{v_4}{R_4} \quad (4)$$

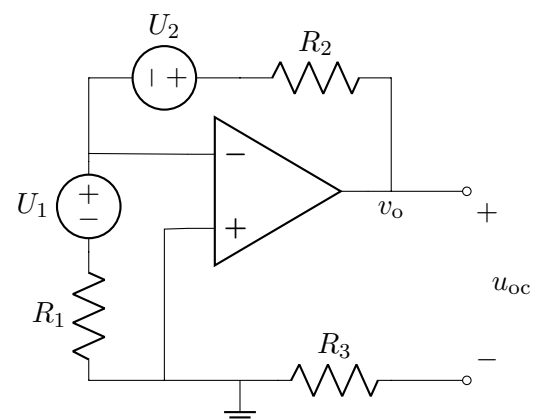
$$\text{KCL}(3): 0 = I + \frac{v_3 - H \frac{v_4 + U_2 - v_3}{R_3}}{R_2} + \frac{v_3 - (v_4 + U_2)}{R_3} \quad (5)$$

The five equations above are all that are needed. Compared to the first method, this method gave fewer equations, by avoiding having three unknown source currents and a marked current. Perhaps the bigger advantage is that only the two final equations (KCL) have to be solved simultaneously: after finding their unknowns v_3 and v_4 the other potentials can be written based on the earlier equations.

Q3

This question could be put in other words as: find the Thevenin (or Norton) equivalent at the marked terminals, for the circuit without R_x , and then find the maximum power that this source can produce.

We're told that the load R_x connected across the terminals has been chosen to obtain this value of power.



First: open-circuit voltage, which is the Thevenin voltage.

What is the opamp's output potential? The non-inverting input is held to zero, so the inverting input must also be at zero, as we have negative feedback and everything's ideal. KCL(out) at the inverting input gives

$$\frac{0 + U_2 - v_o}{R_2} + \frac{0 - U_1 - 0}{R_1} = 0,$$

from which

$$v_o = U_2 - \frac{R_2}{R_1}U_1.$$

In the open-circuit case, there is no current in R_3 , so $u_{oc} = v_o$.

The Thevenin resistance is simply R_3 .

Reason: The potential of the terminal at the opamp output is fixed to whatever is required by the feedback circuit to hold the inverting input to the same potential as the non-inverting input; the opamp is like a voltage source, which has to provide the current for whatever we connect to it, while keeping the necessary potential. If we connected between the zero node and the opamp output, it would appear as an ideal voltage source, with zero resistance. But the lower terminal connects to the zero node through a resistance, R_3 , so this is the Thevenin resistance seen at the terminals.

The Thevenin equivalent at the terminals is therefore

$$U_T = U_2 - \frac{R_2}{R_1}U_1, \quad R_T = R_3.$$

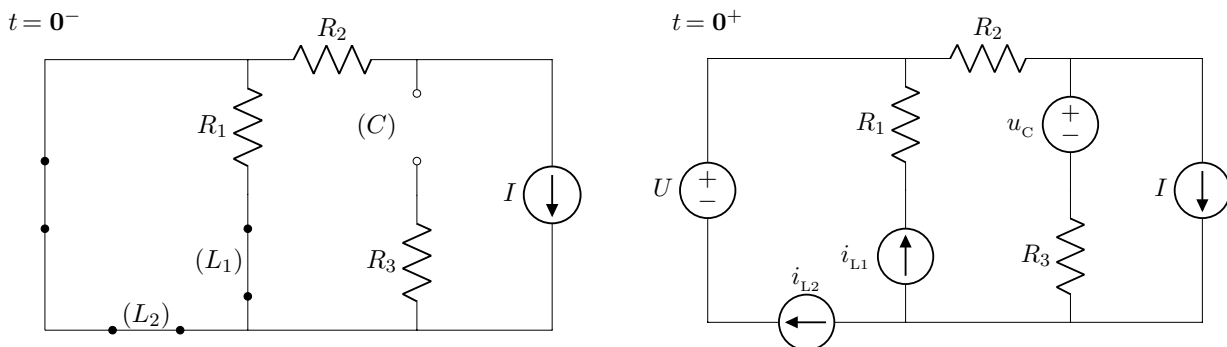
By the usual relations for maximum power, this circuit can supply at its terminals a maximum of

$$P_{\max} = \frac{U_T^2}{4R_T} = \frac{\left(U_2 - \frac{R_2}{R_1}U_1\right)^2}{4R_3}.$$

This maximum power would happen when $R_x = R_3$; but we weren't asked to point that out.

Q4

a) This question concerns the state at $t = 0^+$, just after the step-change of the voltage source. This state is shown on the right below, with inductors and capacitors replaced by current and voltage sources. In order to find the values of these sources, we need to solve for them in the circuit on the left, which shows the circuit at $t = 0^-$, in equilibrium before the step.



Solving the left circuit, with regard to definitions of current and voltage used in the right circuit, we find: $i_{L_2} = I$, $i_{L_1} = 0$, $u_c = -IR_2$. Inserting these in the circuit on the right, and solving:

$P_{R_1}(0^+) = 0$, because $i_{L_1} = 0$, as L_1 has not yet allowed current to flow in this branch.

$P_{R_3}(0^+) = 0$, KCL at the bottom node shows that no current flows in R_3 .

$P_{L_2}(0^+) = I(U - IR_2 - u_c - 0) = UI$, by KVL in the loop of L_2 , U , R_2 , u_c , R_3 .

b) The equilibrium at $t \rightarrow \infty$ differs from $t = 0^-$ only in that the voltage source is now U instead of 0.

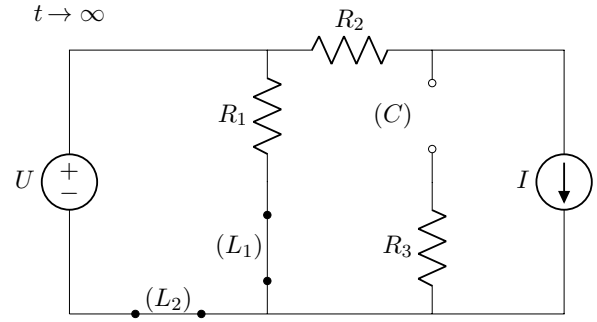
$$W_{L_2}(\infty) = \frac{1}{2}L_2 \left(I + \frac{U}{R_1} \right)^2$$

The current downwards through R_1 is U/R_1 .

By KCL below R_1 , the current in L_2 is therefore $I + U/R_1$.

$$W_C(\infty) = \frac{1}{2}C(U - IR_2)^2$$

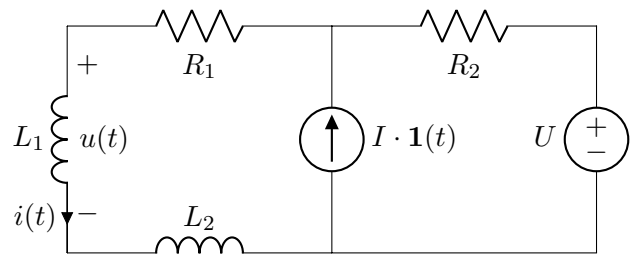
By KVL, the voltage across the capacitor is $U - IR_2$; there is no voltage across R_3 (open circuit capacitor, so by KCL no current, so by Ohm's law no voltage), nor across L_2 (equilibrium).



Q5

The reactive component here is the total of the two inductors. As they are in series, they behave to the rest of the circuit like a single inductor of $L_1 + L_2$.

(This is important: if they were in different places, they might have independent responses with different time-constants, or might interact to give a second-order system. As we can make a single equivalent inductance, we expect this circuit to have the simple behaviour of a first-order system where all quantities are of the form $a + be^{-t/\tau}$.)



During the period of interest ($t > 0$) the current source's value is I . The Thevenin equivalent seen by the two inductors is

$$U_T = U + IR_2, \quad R_T = R_1 + R_2.$$

The Thevenin voltage here is for the upper node (left of R_1) relative to the lower (below I). It can be found from KVL around the open-circuit case, with the inductors not present, after KCL shows that all of I passes through R_2 . The Thevenin resistance can be found by setting the sources (both independent) to zero, and simplifying to a single resistance.

We'll first find the current in the inductors, as that is the continuous quantity. Then we'll find the marked voltage across L_1 from this current.

An inductance behaves as a short-circuit in the equilibrium, so the final current down through L_1 is the short-circuit current of the Thevenin equivalent that was found above: $i(\infty) = \frac{U + IR_2}{R_1 + R_2}$.

Before the step-function, in the equilibrium at $t = 0^-$, the current source is like an open circuit, and the inductors are like short circuits: so in this case the current down L_1 is $i(0^-) = \frac{U}{R_1 + R_2}$. Immediately after the step-function, this current is the same, as current is the continuous quantity in an inductor – it takes energy and time to change it. Therefore, $i(0^+) = \frac{U}{R_1 + R_2}$.

We now have the initial and final values of the current, in the period of interest. The time-constant is found from the Thevenin resistance and the inductance: $\tau = \frac{L_1 + L_2}{R_1 + R_2}$. Using these three results,

$$i(t) = i(\infty) + (i(0^+) - i(\infty)) e^{-t/\tau} = \frac{U + IR_2}{R_1 + R_2} - \frac{IR_2}{R_1 + R_2} e^{-t \frac{R_1 + R_2}{L_2 + L_2}} \quad (t > 0)$$

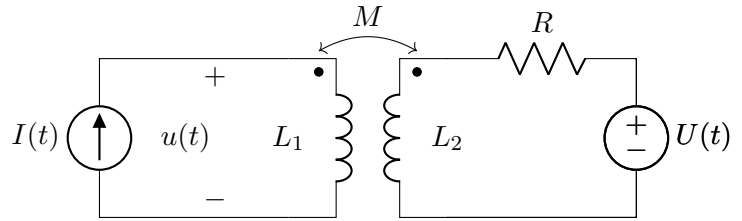
We were actually looking for the voltage $u(t)$ across L_1 , so now we calculate this from the solved current:

$$u(t) = L_1 \frac{di(t)}{dt} = L_1 \frac{R_1 + R_2}{L_1 + L_2} \frac{IR_2}{R_1 + R_2} e^{-t \frac{R_1 + R_2}{L_2 + L_2}} = \frac{IR_2 L_1}{L_1 + L_2} e^{-t \frac{R_1 + R_2}{L_2 + L_2}} \quad (t > 0).$$

Q6

Both sources are sinusoidal, with the same frequency: we can use one solution, representing each source as a phasor.

Let's take $\cos(\omega t)$ as our reference, so that this function corresponds to a phasor with zero angle. Then the $\sin(\omega t)$ function corresponds to an angle of $-\pi/2$.



$$I(t) = \hat{I} \cos(\omega t) \quad U(t) = \hat{U} \sin(\omega t)$$

Now we can express the two sources as phasors at angular frequency ω , as below. (In the later equations, we'll drop the ' (ωt) ' and just write the quantities' symbols I, U, u etc., to make it neater.)

$$I(\omega) = \hat{I}, \quad U(\omega) = -j\hat{U}.$$

Notice that this is the same circuit as Q6 in the previous exam (March 2018), but now we are solving for the voltage across the left coil, rather than for the current in the right coil. One solution method is to start by solving for the current in the right coil, as done in the previous exam's solution, and then use this current, along with $I(\omega)$, to calculate the left coil's voltage u from the mutual inductance equation.

Here, instead, we'll try the method of writing equations for all that we know, and then seeing what we can do with the equations. Let's define the coils' voltages as u_1 (left) and u_2 , with positive reference at the top, and their currents as i_1 and i_2 both defined into the tops of the coils.

We write the equations for the two coils' voltages,

$$\text{mutual inductance (left)} \quad u_1 = j\omega L_1 i_1 + j\omega M i_2 \quad (1)$$

$$\text{mutual inductance (right)} \quad u_2 = j\omega L_2 i_2 + j\omega M i_1 \quad (2)$$

Due to our chosen definitions of the currents and voltages, these equations have no negative signs. The equations of the other parts of the circuit come from KCL in the left loop, and KCL or KVL in the right loop,

$$\text{KCL}_{(\text{left})} \quad i_1 = I \quad (3)$$

$$\text{KVL}_{(\text{right})} \quad u_2 = U - R i_2 \quad (4)$$

We can also note the obvious, that our definition u_1 is the same as the quantity u that we want to find in the end. (The reason for defining u_1 was that it's more consistent with how we've studied mutual inductance calculations, and that experience tells us it's nicer to think about x_1 and x_2 than about x and x_2 .)

Looking at the above equations, we have 4 equations and 4 unknowns: u_1, u_2, i_1, i_2 . Equation (3) directly gives us i_1 . We only want to find u_1 , so we must also eliminate u_2 and i_2 .

Substituting equations (3)(4) into (1)(2),

$$\text{mutual inductance (left)} \quad u_1 = j\omega L_1 I + j\omega M i_2 \quad (5)$$

$$\text{mutual inductance (right)} \quad U - R i_2 = j\omega L_2 i_2 + j\omega M I. \quad (6)$$

Substituting now (6) into (5),

$$u_1 = j\omega L_1 I + j\omega M \frac{U - j\omega M I}{R + j\omega L_2},$$

Now we put in the values of the source phasors,

$$u_1 = j\omega L_1 \hat{I} + j\omega M \frac{-j\hat{U} - j\omega M \hat{I}}{R + j\omega L_2},$$

and ‘simplify’ the result,

$$u_1 = \frac{\omega M \hat{U}}{R + j\omega L_2} + \frac{\omega^2 M^2 \hat{I}}{R + j\omega L_2} + j\omega L_1 \hat{I} = \frac{\omega M \hat{U} + \omega^2 (M^2 - L_1 L_2) \hat{I} + j\omega L_1 R \hat{I}}{R + j\omega L_2}.$$

While (or before!) simplifying, it’s wise to think what we want the result for. In this case, we have to extract the polar form – magnitude and angle – in order to find the time function. That is why the final form, above, was chosen: it has a single rectangular expression on the top and on the bottom, which makes it fairly easy to convert to polar form.

Now we find the magnitude and angle of this phasor. As we’re near the final answer, we’ll call it u instead of the name u_1 that we used for our own working.

$$|u| = \sqrt{\frac{(\omega M \hat{U} + \omega^2 (M^2 - L_1 L_2) \hat{I})^2 + (\omega L_1 R \hat{I})^2}{R^2 + \omega^2 L_2^2}}$$

$$\underline{u} = \tan^{-1} \frac{\omega L_1 R \hat{I}}{\omega M \hat{U} + \omega^2 (M^2 - L_1 L_2) \hat{I}} - \tan^{-1} \frac{\omega L_2}{R}.$$

If you’re being very careful, you might wonder whether the real parts used in the inverse tangents are guaranteed to be non-negative: remember that if a complex number $(a + jb)$ has a negative real part a , its angle is $\tan^{-1} \frac{b}{a} + \pi$. In the expression for u_1 , if $L_1 L_2 > M^2$ and \hat{U} is sufficiently small, the real part on the top could be negative. Due to the relation $M = k\sqrt{L_1 L_2}$, with $0 \leq k \leq 1$, the condition $L_1 L_2 > M^2$ will be true unless $k = 1$. We’ll leave this question/worry just as a point to note if you are programming solutions of this sort of problem.

As we had a cosine reference, the corresponding time function is

$$u(t) = |u| \cos(\omega t + \underline{u}),$$

and we’ll leave it like that instead of inserting the nasty big expressions from above.

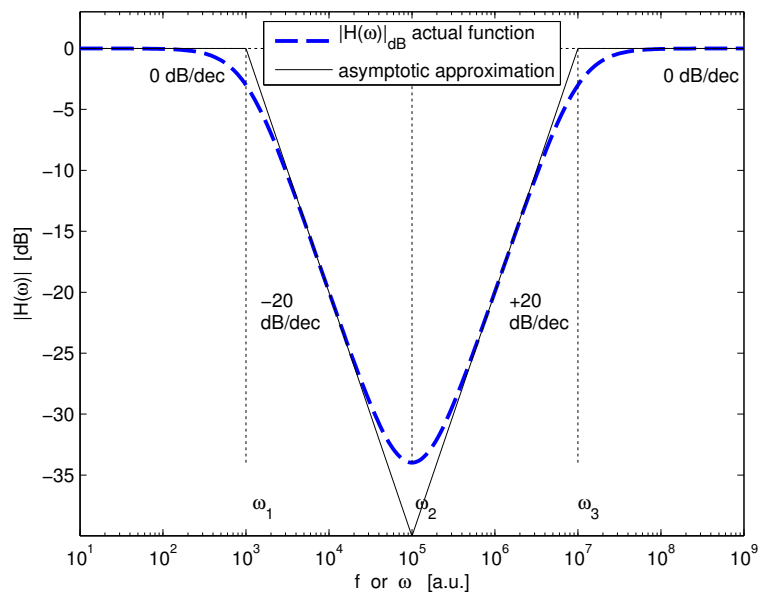
Q7

a)

When $\omega \ll \omega_1$, all the four terms give gain of 1 (0 dB).

When the frequency passes ω_1 , one term in the bottom of the function starts giving a slope -20 dB/decade until ω_2 , at which the two top terms each contribute 20 dB/decade so the total slope is 20 dB/decade.

This continues until ω_3 at which the other bottom term starts to give -20 dB/decade, so that the four terms’ slopes cancel to zero slope for the higher frequencies.

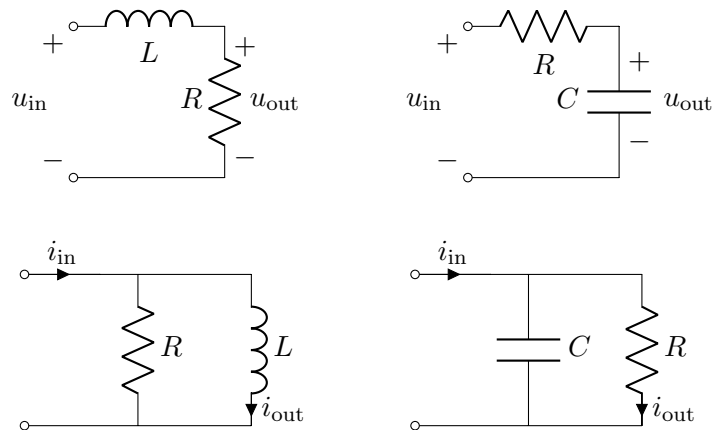


The example shown is only an example! It is a particular case where $\frac{\omega_2}{\omega_1} = \frac{\omega_3}{\omega_2} = 100$. In that case, the amplitude for $\omega \gg \omega_3$ is the same as the amplitude for $\omega \ll \omega_1$, since the negative and the positive gradient both occur over equal ranges of frequency (2 decades).

In other cases, the amplitude for $\omega \gg \omega_3$ could be higher or lower. The position of the lowest point also depends on our choice of how much greater ω_2 is than ω_1 , which we chose to be 100 times.

The only features that must be found in any correct solution are the amplitude of 0 dB at $\omega \ll \omega_1$, and the gradients of ± 20 dB/decade.

b) Some simple circuits with this property of $\frac{u_{out}}{u_{in}} = \frac{1}{1+j\omega/\omega_0}$ are:



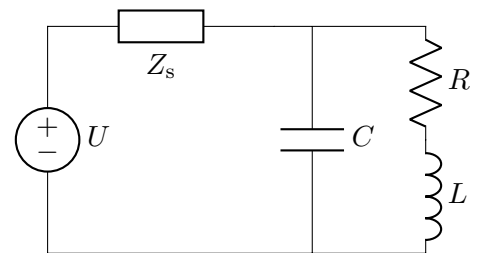
Deriving the network functions, and thereby showing the relation of ω_0 to the component values, is “left as an exercise for the reader”: similar circuits are solved in the notes or exercises of the Topic on time-functions. Since there are only two components in each of these circuits, dimensional analysis is a good cheat-method for finding ω_0 – there isn’t a lot of choice!

Q8

a)

We want to make the complete load (R, L, C) have ‘unity power factor’, which means its complex power will have no reactive part ($Q = 0$) and its total impedance will also be purely real.

One approach is to work with powers. Define the voltage across this load as u :



$$S = \frac{|u|^2}{R - j\omega L} + |u|^2 (-j\omega C),$$

and split this into real and imaginary parts,

$$S = P + jQ = |u|^2 \left(\frac{R}{R^2 + \omega^2 L^2} + j \left(\frac{\omega L}{R^2 + \omega^2 L^2} - \omega C \right) \right).$$

The condition $Q = 0$ requires that

$$\frac{\omega L}{R^2 + \omega^2 L^2} = \omega C, \quad \text{thus} \quad C = \frac{L}{R^2 + \omega^2 L^2}.$$

Another approach – simpler by avoiding defining a quantity u – is to set the total impedance to be real. The two branches are in parallel, so it’s easier to set the total admittance to be real: this is equivalent,

as Z is real if and only if $\frac{1}{Z}$ is real.

$$\Im\left\{j\omega C + \frac{1}{R + j\omega L}\right\} = 0, \quad \text{thus} \quad C = \frac{L}{R^2 + \omega^2 L^2}.$$

b) We want to maximise the active power delivered to the ‘load’. The ac maximum power theorem is appropriate to a fixed source (impedance and open-circuit voltage) and a load that can be varied to give whatever magnitude and phase-angle of current are necessary to obtain maximum power from the source.

We often have a choice about which parts of the circuit to consider as the load, and which as the source. It’s essential that the load contains the components where the active power is to be consumed. Reactive components could be put in the load or the source, but it generally is nice and clear if all the parts that we are free to choose are in the load. That’s easy in this case, if we treat R , L and C as the load.

For maximum power, the condition is $Z_{\text{load}} = Z_{\text{source}}^*$. Thus,

$$Z_s^* = \frac{1}{\frac{1}{R + j\omega L} + j\omega C} = \frac{R + j\omega L}{(1 - \omega^2 L^2) + j\omega CR},$$

which is a nasty mixture of real and imaginary parts and multiple occurrences of the unknown R .

It’s probably better to start again, expressing the equation in terms of *admittances*,

$$\frac{1}{Z_s} = \left(\frac{1}{R + j\omega L} + j\omega C\right)^*,$$

from which, taking real and imaginary parts of Z_s ,

$$\begin{aligned} \frac{R}{R^2 + \omega^2 L^2} &= \Re\left\{\frac{1}{Z_s}\right\} \\ C &= \frac{L}{R^2 + \omega^2 L^2} - \frac{1}{\omega} \Im\left\{\frac{1}{Z_s}\right\}. \end{aligned}$$

Having solved the first of these equations to get R , the second is easier.

This question, Q8b, was so open that it’s not a demand to do more than showing the initial relation that equates a correct expression for load impedance with the conjugate of the source impedance. If this is clear, it should give the full points. Perhaps if someone does a really clever job of going further, they might get a little bonus.

Q9

We’ll start by finding the loads’ active and reactive powers, from the two given quantities for each load.

Source: line-voltage U , angular frequency ω .

Load 1: $|S_1|$, PF_1 :

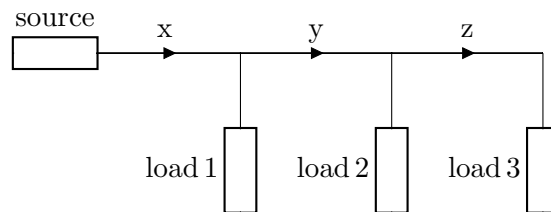
$$P_1 = |S_1| \text{PF}_1, \quad Q_1 = |S_1| \sqrt{1 - \text{PF}_1^2}.$$

Load 2: P_2 , PF_2 :

$$P_2 = P_2, \quad Q_2 = P_2 \sqrt{\frac{1}{\text{PF}_2^2} - 1}.$$

Load 3: $|S_3|$, θ_3 .

$$P_3 = |S_3| \cos \theta_3, \quad Q_3 = -|S_3| \sin \theta_3.$$



The reason for the negative sign on the $\sin \theta_3$ term is that this angle was defined as current relative to voltage (e.g. for each phase of the load), meaning that a lagging load (one that consumes reactive power) would have a negative angle. As we are defining power flow into the load, we want a lagging load to have a positive value of reactive power.

Note that each line in the diagram indicates a three-phase connection. As the source and all the loads are balanced, it does not matter whether there is a neutral or not, or whether the loads have a star or delta connection. This ‘single-line diagram’ for showing a three-phase system is a common way to simplify diagrams; it has been shown in the course-notes for the Topic on balanced three-phase calculations, and used in an exercise in the examples.

a) At point ‘y’, the power flow is the sum of powers of loads 2 and 3. Summing the reactive powers calculated above,

$$Q_y = Q_2 + Q_3 = P_2 \sqrt{\frac{1}{\text{PF}_2^2} - 1} - |S_3| \sin \theta_3.$$

b) At point ‘x’, the sum of all the loads is flowing. We know that in a balanced three-phase system the apparent power flow is given by $|S| = \sqrt{3}UI$, where U and I are line voltage and line current magnitudes. We can find the apparent power from the sums of P and Q , and we know the line-voltage magnitude, so we can use this equation:

$$\begin{aligned} I &= \frac{|S|}{\sqrt{3}U} = \frac{\sqrt{(\sum P)^2 + (\sum Q)^2}}{\sqrt{3}U} \dots \\ &= \frac{\sqrt{\left(|S_1| \text{PF}_1 + P_2 + |S_3| \cos \theta_3\right)^2 + \left(|S_1| \sqrt{1 - \text{PF}_1^2} + P_2 \sqrt{\frac{1}{\text{PF}_2^2} - 1} - |S_3| \sin \theta_3\right)^2}}{\sqrt{3}U}. \end{aligned}$$

Thought: I really don’t like the double symbol ‘PF’ for power factor in equations. How about ‘f’ or ‘k’? Likewise, $|S|$ for apparent power and S for complex power is nice and simple when mainly dealing with complex power, but it would be nicer to have something like S and \mathbf{S} to avoid lots of $||$ symbols in equations. The reason for our choice in the course is the relative difficulty of drawing bold-font on a board, or the tedium of always needing an overbar (\bar{S}) if instead the bar method is used for showing complex quantities, given that most of the quantities in this part of the course *are* complex.

c) This task is to find the angle of the current in line-b at point ‘z’, which means the current going to just load 3. The reference angle is given by defining the angle of the voltage u_{ab} as being 30° .

The solution is $\alpha_{i_b} = -120^\circ + \theta_3$, or in radians $-\frac{2\pi}{3} + \theta_3$.

One method. Imagine there to be a neutral, and that we define the line potentials relative to this: v_a , v_b , v_c . If we draw out the three potentials in the complex plane at angles such that the line from v_b to v_a (which shows the direction of u_{ab}) has an angle of 30° to the real axis, we note that $\angle v_a = 0^\circ$ and so $\angle v_b = -120^\circ$. Assume the load to be star-connected. As the load is balanced, each phase has similar properties (e.g. the same angle between its current and voltage). As the angle of current relative to voltage is stated to be θ_3 for this load, the phase current must be at θ_3 relative to the phase voltage. For the phase connected to line 3, the phase voltage is -120° , so the phase current is at the angle $-120^\circ + \theta_3$. As a star-connection has the same phase current as the line current, this is the solution.

Another method. This starts directly from the given angle, without considering any neutral point or star connection. We know that the angles are defined such that the voltage of ‘a’ relative to ‘b’ is 30° ; then with phase-rotation a,b,c, the voltage of ‘b’ relative to ‘c’ will be shifted -120° further, to -90° . If we assume a delta load with phase-impedances Z , the current in line ‘b’ in the direction of the arrow at point ‘z’ is:

$$i_b = \frac{u_{ba}}{Z} + \frac{u_{bc}}{Z} = \frac{-u_{ab}}{Z} + \frac{u_{bc}}{Z} = \frac{U}{Z} \left(1 \angle -150^\circ + 1 \angle -90^\circ\right) = \sqrt{3} \frac{U}{Z} \angle -120^\circ.$$

This is not the final answer: U is a real number, describing the line voltage magnitude, but Z is the impedance in each phase of the load. The load is balanced and has an angle of θ_3 of the current relative

to the voltage in each phase. An impedance relates a current (i) and voltage (u) according to $Z = \frac{u}{i}$. From this, it must be in our case that $\angle Z = -\theta_3$ in order that $Z = \frac{|u|\angle\alpha}{|i|\angle\alpha + \theta_3}$, where α is whatever angle the voltage across a particular phase is defined as having. Putting this in to find the answer,

$$\angle i_b = \angle \sqrt{3} \frac{U}{Z} \angle -120^\circ = \angle U - (-\theta_3) - 120^\circ = \theta_3 - 120^\circ.$$

In these examples we note how assumptions about a particular load structure (star or delta) might help us to think, and they come to the same conclusion about what happens in the system outside the load.