KTH EI1120 Elkretsanalys (CENMI) KS 1 — 2020-01-30 kl. 08–10

Permitted material: Beyond writing-equipment, a single piece of paper up to A4 size can be brought, with free choice of content: handwritten, printed; small, large; text, diagram, image; one or both sides, etc. This paper does not need to be handed in with the exam.

Unless it is stated otherwise, the final answer to a question should be expressed in terms of the known quantities given in the question, and any clear simplifications should be done. Component values such as R for a resistor, U for an independent voltage source, or K for a dependent source, are assumed to be known quantities. Marked currents or voltages such as i_x are assumed to be definitions, not known quantities.

Clearly drawn and labelled diagrams are a good way to help yourself avoid mistakes, and to make clear to others what you are doing. By showing clearly your intermediate steps in a solution, you improve your chance of getting points even if the final result is wrong. You may write in Swedish or English; but we suggest that writing in either is seldom necessary if you make good use of diagrams and equations!

KS1 does not give any direct grade. Its points will be used to replace Section-A in the final exam or re-exam, if this would improve your points there. See therefore the rules for the exam to relate the points to grades: at least 40% is needed in Section-A alone, as well as 50% overall.

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1) [4p]

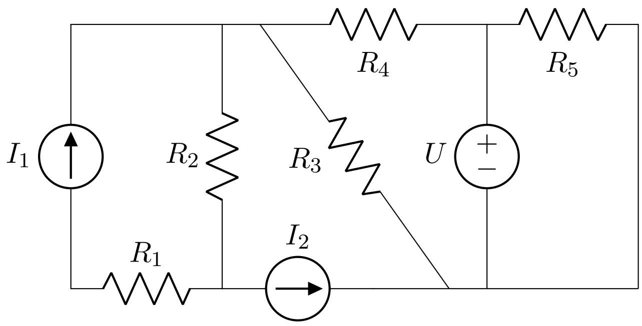
Determine:

a) [1p] the power into R_5

b) [1p] the power into R_2

c) [1p] the power out of I_1

d) [1p] the power out of U



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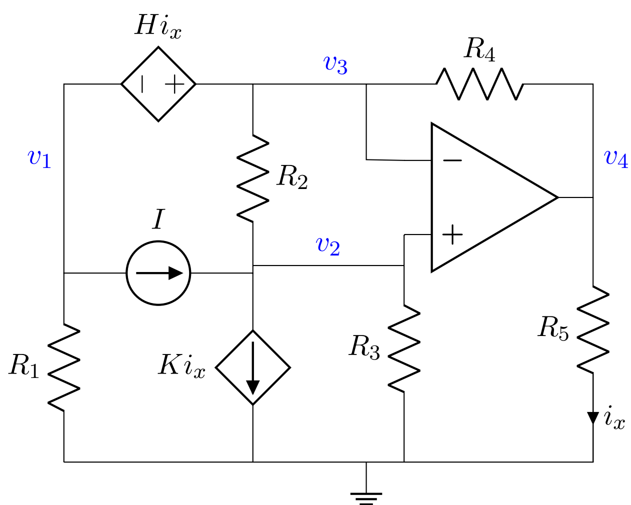
2) [4p]

Write equations that could be solved to find the node potentials v_1, v_2, v_3, v_4.

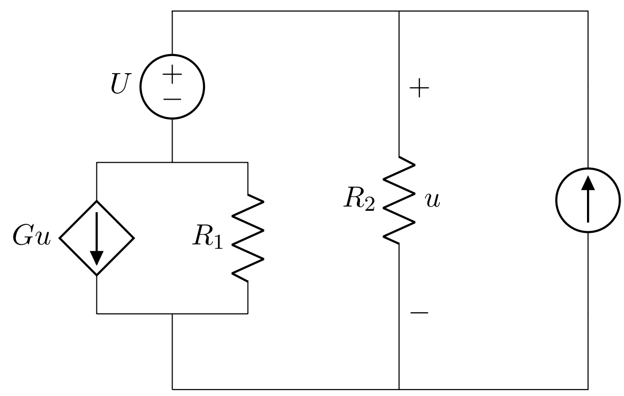
You do not need to solve the equations, nor to write them in the most compact way. But the equations must be sufficient to solve for the potentials, in terms of component values, without further information (i.e. without needing to look at the circuit diagram).



3) [4p]

What value should the independent current source at the right of this circuit have, in order that it absorbs (receives) the highest possible power?

Express this value in terms of the values of the other components in the circuit.



Översättningar:

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text, diagram, bild; stor eller liten textstorlek, o.s.v. Det måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas.

Var tydlig med diagram och definitioner av variabler. Du får skriva på svenska eller engelska, men vi rekommenderar att diagram och ekvationer används i stället i de flesta fall.

KS1 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-A i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

1. Bestäm följande storheter:

a) [1p] effekten absorberad av R_5

b) [1p] effekten absorberad av R_2

c) [1p] effekten levererad från I_1

d) [1p] effekten levererad från U.

2. Skriv ekvationer som skulle kunna lösas för de markerade potentialerna v_1, v_2, v_3, v_4.
Du måste inte lösa ekvationerna, men de måste gå att lösa utan mer information om kretsen.

3. Vilket värde måste den oberoende strömkällan till höger i kretsen ha för att den ska absorbera den högsta möjliga effekten från resten av kretsen? Uttrycka värdet som funktion av andra komponentvärden.

The End. *Don't waste remaining time . . . check your solutions!*

Solutions (EI1120 KS 1 VT20, 2020-01-30)

Q1.

a) $P_{\text{R5}} = \dfrac{U^2}{R_5}$.

R_5 is in parallel with source U, which therefore fixes the voltage across it.
In other words: KVL around the rightmost loop!
The power into a resistor R that has voltage u across it is $ui = u \cdot u/R = u^2/R$; this is a result that's useful for finding power from just the voltage. The definition direction doesn't matter, as the voltage is squared: the resistor must always absorb power when a current passes through it in either direction.

b) $P_{\text{R2}} = (I_1 + I_2)^2 \, R_2$.

KCL below R_2 gives the current down R_2 as $I_1 + I_2$.
Similar to the expression in part 'a', the power into a resistor R with current i through it is $ui = i \cdot iR = i^2 R$, leading to the above result for P_{R2}.

c) $P_{\text{I1}} = R_1 I_1^2 + R_2 I_1 \left(I_1 + I_2\right)$.

In order to find the power out of the current source I_1 we should find the voltage across it, and multiply that by its current (with suitable direction/sign). If we define the voltage as u, with its reference (+) where the current-source's current is marked as coming out, then $P_{\text{I1}} = uI_1$.

KVL in the leftmost loop would give us u, as long as we know the voltages across the resistors. In this circuit, those voltage are easily found by KCL and Ohm's law: $I_1 + I_2$ down through R_2, and I_1 along to the left through R_1. Thus $u = (I_1 + I_2)\, R_2 + I_1 R_1$. Multiplying this result by u, the final result above is obtained.

d) $P_{\text{U}} = U^2 \left(\dfrac{1}{R_3 + R_4} + \dfrac{1}{R_5} \right) + U I_2 \dfrac{R_3}{R_3 + R_4}$.

As in 'c' we find a source's output power by finding whichever of its two quantitites isn't known, and multiplying with the known one. This source has a known voltage U. Define its current as i out from the + side of U.

From where the voltage source is connected, the three components furthest left in the circuit are hidden behind the current source I_2, since this source fixes the current that passes from their region to the right of the circuit. The whole branch of I_1, I_2, R_1, R_2 can be replaced by just a source I_2, which is in parallel with R_3, and this combination is in series with R_4. Now I_2 and R_3 are Norton source that can be transformed to a Thevenin and combined with R_4, to give a Thevenin source with voltage $-I_2 R_3$ (if reference upwards) and resistance $R_3 + R_4$.

The current from the voltage source U is then the sum of its currents to this Thevenin source on the left, and the resistor on the right: $i = \frac{U - -I_2 R_3}{R_3 + R_4} + \frac{U}{R_5} = \frac{U}{R_3 + R_4} + \frac{U}{R_5} + \frac{I_2 R_3}{R_3 + R_4}$. Multiplying this by U leads to the expression at the top.

Note: In this part 'd' in particular, valid solutions by different methods could look quite different and yet be equivalent.

Q2.

Two solution methods are shown below, following the systematic methods from the course. The question doesn't demand these methods to be used, although it's usually prudent to do so except in a very simple case! So there are many styles of solution that could be correct ... or that might be subtly *incorrect* if you didn't carefully follow a 'safe' method!

I. Extended nodal analysis ("the simple way")

Voltage sources' currents aren't known initially and usually aren't easily expressed in terms of other components before the solution is done. So we *define* these currents and use them in the solution.

For the dependent voltage source Hi_x, we'll define the unknown current as as i_α into its '+' terminal.

At the opamp output, let's define i_o inward. (The opamp output is treated as a voltage source, whose other side is connected to the reference node.)

First we write KCL at all nodes except the reference.[1]

$$\mathrm{KCL}(1)_{\mathrm{(out)}}: \quad 0 = \frac{v_1}{R_1} + I - i_\alpha \tag{1}$$

$$\mathrm{KCL}(2)_{\mathrm{(out)}}: \quad 0 = Ki_x - I + \frac{v_2 - v_3}{R_2} + 0 + \frac{v_2}{R_3} \tag{2}$$

$$\mathrm{KCL}(3)_{\mathrm{(out)}}: \quad 0 = \frac{v_3 - v_2}{R_2} + i_\alpha + 0 + \frac{v_3 - v_4}{R_4} \tag{3}$$

$$\mathrm{KCL}(4)_{\mathrm{(out)}}: \quad 0 = \frac{v_4 - v_3}{R_4} + i_\mathrm{o} + \frac{v_4}{R_5} \tag{4}$$

These are only 4 equations so far, but with 6 unknowns: v_1, v_2, v_3, v_4, i_α, i_o.

Now we add the further information given by the voltage sources, as each voltage source provides an equation as well as an extra unknown current. In this circuit there's just the one voltage source, which is the dependent source Hi_x:

$$v_3 - v_1 = Hi_x. \tag{5}$$

The above equation introduced a further unknown, i_x, which reminds us that we need to define the marked (but unknown) quantities controlling any dependent sources in the circuit:

$$i_x = \frac{v_4}{R_5}. \tag{6}$$

Now there are 6 equations and 7 unknowns. The remaining information that's not yet expressed in the equations is the relation that the opamp introduces:

$$v_2 - v_3 = 0. \tag{7}$$

Now the above set of 7 equations in 7 unknowns should be solvable. Our confidence in this comes from having followed the systematic way of writing the equations, which avoids writing linearly dependent equations (a real danger), and also from the belief that the circuits we're given in exams are well-behaved ones without internal contradictions!

[1] About the KCL: Zero currents at opamp inputs are included in these equations to show that we've considered every terminal connected to the respective node; but this isn't required in your solutions. It also follows from our knowledge of ideal opamps (with negative feedback) that no current will flow in R_2 because of the opamp inputs having the same potential: however, we'll follow here the classic 'extended method' where we don't worry about simplifying but just write down what the circuit tells us as equations and leave the thinking to when the equations are solved!

II. Nodal analysis by simplifications including supernodes

Here's an alternative way to solve it, again by writing equations, but now trying to simplify the equations "before they're written", rather than leaving that for later.

There are 5 nodes in total, one of these being the reference node.
The opamp output terminal connects to node 4 (where v_4 is marked), so node 4 is connected by a (hidden) voltage source to the reference node. Node 4 can therefore be treated as part of a supernode: the supernode contains the reference node, so the rule of not writing KCL at any part of the reference supernode[2] tells us not to write KCL at node 4, which lets us avoid the unknown current in the opamp output terminal.

Of the remaining nodes, 2 are connected by the voltage source Hi_x, so they form one supernode. The nodes within this supernode have potentials related by $v_1 = v_3 - Hi_x$. We choose just *one* of the potentials in the supernode to use in the KCL equations that we'll have to solve. Let's choose v_3, since that node appears to connect to more terminals. However, in order to write KCL for resistors at node 1, and to fulfil the question's requirement of "equations that can be solved for all potentials", we must also use the equation $v_1 = v_3 - Hi_x$ in KCL and the final answer, to show how to find v_1 from v_3. We'd like to avoid introducing a new unknown i_x in the equations, so we instead look at where it's marked in the circuit, and try to express it in terms of known quantities or unknown quantities that we already have in our equations: thus, $i_x = v_4/R_5$. The relation between v_1 and v_3 is then written as

$$v_1 \quad = \quad v_3 - \frac{H}{R_5} v_4. \tag{1}$$

The opamp gave us an unknown potential v_4 and stopped us writing KCL at that node! But it compensates for that by giving us a relation about its inputs: $v_3 = v_2$. So only one of these quantities needs to be used as an unknown in our equations. We've earlier chosen to use v_3, so we'll write v_3 instead of v_2 if v_2 arises in our equations.

We should also mention this in the solution, to allow v_2 to be found,

$$v_2 \quad = \quad v_3. \tag{2}$$

The KCL equations we need to write are the follow two, at the node of v_2 and at the supernode that doesn't contain the reference. (Don't be confused by $v_3 - v_3$ – remember that this comes from $v_3 - v_2$ or vice versa, and we could have simplified it to zero before writing it.)

$$\text{KCL(2)}_{(\text{out})}: \quad 0 \quad = \quad K\frac{v_4}{R_5} - I + \frac{v_3 - v_3}{R_2} + 0 + \frac{v_3}{R_3}. \tag{3}$$

$$\text{KCL(3,4)}_{(\text{out})}: \quad 0 \quad = \quad \frac{v_3 - \frac{H}{R_5}v_4}{R_1} + I + \frac{v_3 - v_3}{R_2} \tag{4}$$

Now we have 4 equations in total, for the 4 unknown potentials that we're required to solve. Unlike the equations in the extended method, these ones are of two distinct types: the KCL equations have just the right number of unknowns to be solved by themselves, and then the solutions for v_3 and v_4 can be put back into the earlier two equations that tell us how to find v_1 and v_2 from these. That's nicer than the equations from the extended method, that require more thought about where to start. However, if we preferred algebra to circuits, we might prefer to rearrange and substitute the extended-method equations to become the supernode-based equations, instead of using a circuit-based method to write

[2] *Supernode: opamp output and reference.* Note that this case with an opamp has a difference compared to when an independent voltage source connects from the reference node to another node. A node that is connected to the reference through an independent source can have its potential written directly in terms of the source value. This is still "sort-of true" for the usual sort of dependent voltage source, which would allow potentials to be related by its value and controlling variable, such as Hi_x. However, in the case of the opamp, the 'known' thing it gives us ($v_+ = v_-$) concerns other nodes than its outpu, so we're left with an initially unknown voltage at the output, and we have to keep this unknown potential v_4. That's why we have the unusual situation of needing to keep an unknown potential at a node that's part of the reference supernode.

the supernode equations.

III. Non-systematic solution: simplifications etc

Sometimes a circuit can be solved by a nice step-by-step solution, perhaps using some of the simplifications or theorems we've studied.

However, the present circuit doesn't give us any really nice way that doesn't largely repeat the steps from the supernode-based systematic solution. There is only one independent source, so superposition isn't useful. There is also interdependence of variables, which prevents solving just one part at a time: the potentials v_2 and v_3 depend on i_x due to the dependent sources, and i_x depends on the opamp output potential, which in turn depends on v_2 and v_3, and so requires solution of simultaneous equations.

Q3.

The value of the independent current source should be

$$I_{\text{max-power}} = \frac{-U}{2R_1}$$

in order for this source to absorb from the other components the maximum possible power (this is on the assumption that the other components' values are fixed, and just this source's value can be varied).

A derivation.

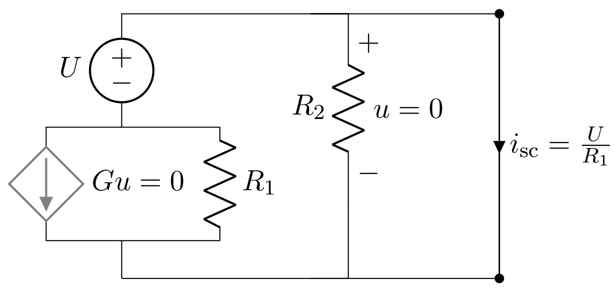
For a linear, two-terminal dc circuit the relation of u and i at its terminals is a straight line, so such a circuit can be represented equivalently by a Thevenin or Norton source. We have seen (maximum-power topic) that such a circuit can deliver its maximum power from its terminals when the current is half of the short-circuit current; or, equivalently, when its voltage is half of the open-circuit voltage.

The circuit in this question can be regarded as two parts, each with two terminals, with these parts connected together. One part is the unknown current source whose value we must determine. The other part is all the other components. This choice of splitting the circuit is useful, because it puts all the fixed values such as U, R_2 etc into the part whose maximum power is to be found. Then we just have to choose the current source value to be the right one to obtain this maximum power from the rest of the cicruit.

We have to find a current source's value, and we know that this can be found as half of the short-circuit current. It is therefore most convenient to find the short-circuit current of the other part of the circuit. There is then no need to consider the Thevenin resistance of the circuit, or to find the open-circuit voltage. It also turns out that finding the short-circuit current is relatively easy in this case.

The circuit on the right is the original one with its current source replaced by a short-circuit in order to find the short-circuit current i_{sc} of this "other part" of the circuit.

The short circuit forces $u = 0$, and thus also $Gu = 0$. This makes the circuit much simpler, as no current flows in R_2 or in the dependent source.



Thus, the short-circuit current is $i_{\text{sc}} = U/R_1$, based on taking KVL around R_1, U and the short-circuit.

Now we remove the short-circuit and put back the current source as in the original circuit, with its arrow pointing up. In order to have $\frac{1}{2}i_{\text{sc}}$ flowing at the terminals, so that the maximum power that the rest of the circuit can provide will go to the independent current source, the source value must be $-\frac{1}{2}i_{\text{sc}}$.

The same solution can be reached by longer routes such as:

- Find the source resistance also, but then realise you don't need it.
- Find the open-circuit voltage and resistance, and find the short-circuit current from their ratio.
- Write an equation for the relation of current and voltage at the terminals, and determine short-circuit current, resistance or open-circuit voltage from this.
- Derive an expression for the power into the independent current source as a function of its value I, then differentiate to find the maximum, thus not not taking advantage of Thevenin/Norton or maximum-power theorems.

To demonstrate one: "write an equation" for the u-i relation at the terminals of the circuit (the circuit other than the unknown source).

Define the terminal voltage as u and the terminal current as i, both unknown, with directions as implied in the above diagram. (The current is marked i_{sc} in that diagram as there is a specific thing connected to the terminals – a short-circuit. But in this new case we just consider "something" unknown at the terminals.)

Write KCL at the top:

$$i + \frac{u}{R_2} + \frac{u-U}{R_1} + Gu = 0,$$

whence

$$u\left(\frac{1}{R_1} + \frac{1}{R_2} + G\right) = \frac{U}{R_1} - i.$$

or

$$u\,\frac{R_1 + R_2 + R_1R_2G}{R_1R_2} = \frac{U}{R_1} - i.$$

Now rearrange into a nice form that separates the u, i and constant parts. One of these forms directly describes a Thevenin equivalent,

$$u = \frac{UR_2}{R_1 + R_2 + R_1R_2G} - \frac{R_1R_2}{R_1 + R_2 + R_1R_2G}\,i = U_{\mathrm{T}} - R_{\mathrm{T}}i.$$

Another directly describes a Norton equivalent,

$$i = \frac{U}{R_1} - \frac{R_1 + R_2 + R_1R_2G}{R_1R_2}u = I_{\mathrm{N}} - R_{\mathrm{N}}u.$$

I don't know why we bother using separate names for R_{T} or R_{N}, given that they're the same thing; but it's become a habit now, so I might cause confusion if I break it.

From the Norton form the short-circuit current is clear as $\frac{U}{R_1}$.

From the Thevenin form it is found by simplifying $U_{\mathrm{T}}/R_{\mathrm{T}}$.