KTH EI1120 Elkretsanalys (CENMI) KS 2 — 2020-02-12 kl. 08–10

Permitted material: Besides writing-equipment, up to two pieces of paper up to A4 size can be brought, with free choice of content: handwritten, printed; small, large; text, diagram, image; one or both sides, etc. These papers do not need to be handed in with the exam.

Unless it is stated otherwise, the final answer to a question should be expressed in terms of the known quantities given in the question, and any clear simplifications should be done. Component values such as R for a resistor, U for an independent voltage source, or K for a dependent source, are assumed to be known quantities. Marked currents or voltages such as i_x are assumed to be definitions, not known quantities.

Clearly drawn and labelled diagrams are a good way to help yourself avoid mistakes, and to make clear to others what you are doing. By showing clearly your intermediate steps in a solution, you improve your chance of getting points even if the final result is wrong. You may write in Swedish or English; but we suggest that writing in either is seldom necessary if you make good use of diagrams and equations!

KS2 does not give any direct grade. Its points will be used to replace Section-B in the final exam or re-exam, if this would improve your points there. See therefore the rules for the exam to relate the points to grades: at least 40% is needed in Section-B alone, as well as 50% overall.

Nathaniel Taylor (08 790 6222)

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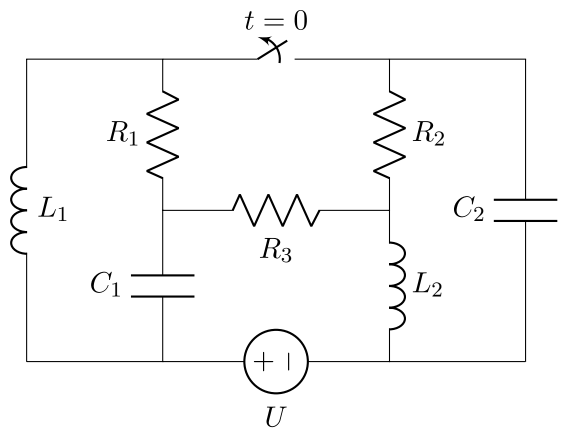
1) [5p] Find:

a) [1p] The power absorbed by R_2 at $t = 0^-$.

b) [1p] The energy stored in C_1 at $t = 0^+$.

c) [2p] The power supplied by L_2 at $t = 0^+$.

d) [1p] The power supplied by source U as $t \to \infty$.



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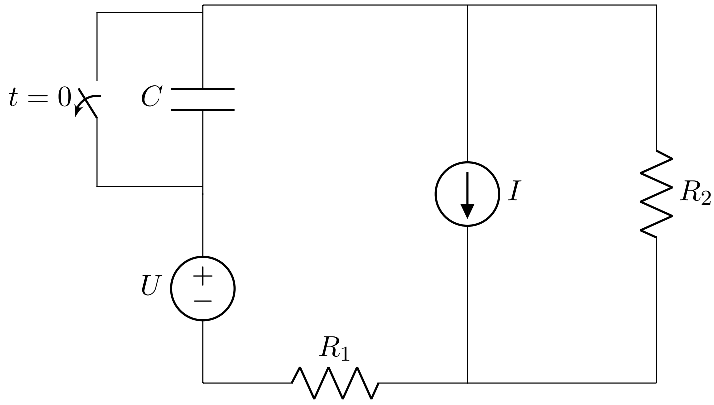
2) [5p]

The switch opens at $t = 0$.

Find the power delivered by the current source I, as a function of time for $t > 0$.



Översättningar:

Hjälpmedel: Upp till två A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text, diagram, bild; stor eller liten textstorlek, o.s.v. Dessa måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas.

Var tydlig med diagram och definitioner av variabler. Du får skriva på svenska eller engelska, men vi rekommenderar att diagram och ekvationer används i stället i de flesta fall.

KS2 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-B i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

1. Bestäm följande storheter:

a) [1p] Effekten absorberad av R_2 vid $t = 0^-$.

b) [1p] Energin lagrad i C_1 vid $t = 0^+$.

c) [2p] Effekten levererad från L_2 vid $t = 0^+$.

d) [1p] Effekten levererad av källan U vid $t \to \infty$.

2. Brytaren öppnas vid tid $t = 0$.

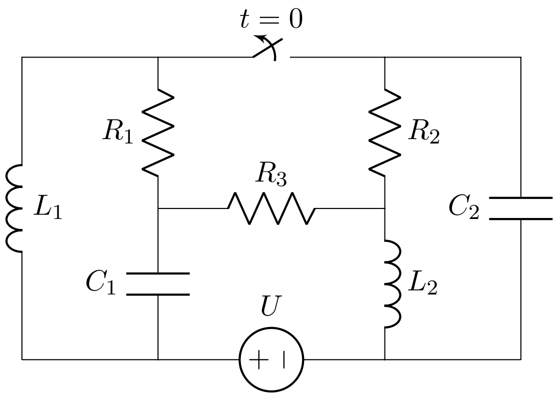
Bestäm effekten levererad från strömkällen I, för $t > 0$.

The End. *Don't waste remaining time . . . check your solutions!*

Solutions (EI1120 KS 2 VT20, 2020-02-12)

Q1.

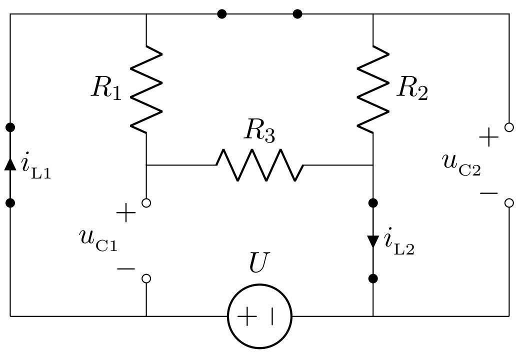
The original circuit is shown on the right.



The first part-question is about a value at $t = 0^-$, and the next two parts are about $t = 0^+$ for which we need to find values from $t = 0^-$ anyway.

So we consider first the time $t = 0^-$, when the switch is still closed. First we re-draw the circuit in the simplest way for this specific time, as shown in the next diagram, below.

Initial Equilibrium: $t = 0^-$



This is an equilibrium, as the circuit shows no change at times other than $t = 0$. Inductors can be represented as short-circuits and capacitors as open-circuits. We mark the continuous variables of the inductors and capacitors, in whatever direction we feel like.

After some thought and perhaps further re-drawing, the state at $t = 0^-$ is seen to be a parallel connection of three branches: one is the source U, one is R_2, and the other is the series combination of R_1 and R_3.

By KVL around the loop of U, L_1, R_2, L_2, the voltage across R_2 is found as U. Therefore,

$$\text{Solution of subquestion } \mathbf{a)} \quad P_{R2} = U^2/R_2.$$

We're only asked to find the above answer at $t = 0^-$, but it's clear we're going to need some others for solving later questions at $t = 0^+$, so let's just find all the continuous quantities of the inductors and capacitors:

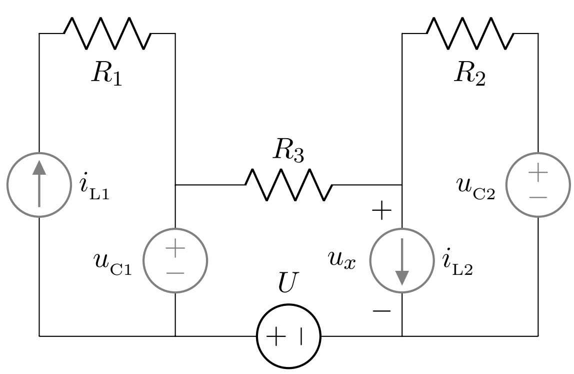
$i_{L1}(0^-) = U\left(\frac{1}{R_2} + \frac{1}{R_1+R_3}\right)$ equivalent resistance of the two resistor-branches

$i_{L2}(0^-) = i_{L1}(0^-)$ KCL at each side of the source, or at both sides together

$u_{C1}(0^-) = \frac{-UR_1}{R_1+R_3}$ voltage-division R_1 & R_3, and KVL around leftmost loop

$u_{C2}(0^-) = U$ KVL around outer loop

Immediately after the change: $t = 0^+$



We can re-draw the circuit again for this time, now with the switch open.

Continuity implies that at $t = 0^+$ the continuous quantities we found for the inductors and capacitors at $t = 0^-$ will not yet have changed.

The inductors can be represented by current sources, and the capacitors by voltage sources, with values according to what was shown above at $t = 0^-$.

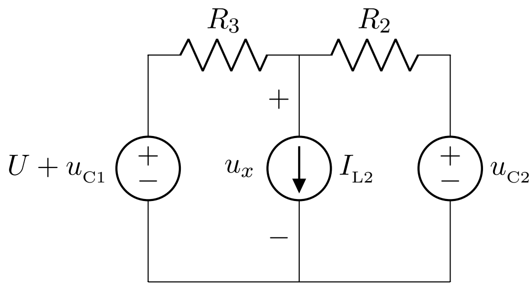
The energy associated with each continuous quantity is the same as at $t = 0^-$.
The answer to part 'b' is therefore

$$\text{Solution of subquestion } \mathbf{b)} \quad W_{\mathrm{C1}} = \frac{1}{2}C_1 u_{\mathrm{C1}}(0^+)^2 = \frac{1}{2}C_1 u_{\mathrm{C1}}(0^-)^2 = \frac{1}{2}C_1\left(\frac{UR_1}{R_1+R_3}\right)^2.$$

A voltage u_x has been marked across the position of L_2. The power delivered from L_2 is the product of its current i_{L2} and voltage u_x, with suitable choice of sign. For our choice of direction, the sign should be negative, as we want to find the power out, but the current is marked into the +-defined side of the voltage.

So now we must determine u_x in order to solve subquestion 'c'. It might help to think of potentials. There are six nodes in total (see the circuit above, for $t = 0^+$). If the one below the marked voltage u_x is taken as the reference, then three of the remaining five nodes have potentials fixed by the voltage sources. The node above L_2 has unknown potential, which is the value defined as u_x, which we want to find. (The node above L_1 also has unknown potential, but it could be easily found as the current in R_1 is fixed by L_1. And this potential won't affect the solution for u_x as the potential above C_1 is fixed by voltage sources, so the current in R_3 doesn't depend on the potential above L_1.)

The relevant part to solve for u_x is therefore as shown on the right. Components U and C_1 have been combined to a single source.



KCL at the top-middle node, or the bottom node, gives:

$$\frac{u_x - U - u_{\mathrm{C1}}}{R_3} + \frac{u_x - u_{\mathrm{C2}}}{R_2} + i_{\mathrm{L2}} = 0,$$

from which

$$u_x = \frac{1}{\frac{1}{R_2}+\frac{1}{R_3}}\left(\frac{U+u_{\mathrm{C1}}}{R_3} + \frac{u_{\mathrm{C2}}}{R_2} - i_{\mathrm{L2}}\right) = \frac{R_2U + R_2u_{\mathrm{C1}} + R_3u_{\mathrm{C2}} - R_2R_3i_{\mathrm{L2}}}{R_2+R_3}.$$

One alternative method to arrive here would be source-transformation of the two Thevenin parts, to give three parallel current sources and two parallel resistors. The first form of the expression for u_x, above left, strongly hints at this, as it is a product of two parallel resistances and three currents.

Up to here we've used our own definitions such as u_{C2}, as they are short and neat. Now we must put in the actual values in terms of known quantities, based on what was found at $t = 0^-$:

$$u_x = \frac{R_2U - \frac{R_1R_2U}{R_1+R_3} + R_3U - R_2R_3U\left(\frac{1}{R_2}+\frac{1}{R_1+R_3}\right)}{R_2+R_3}.$$

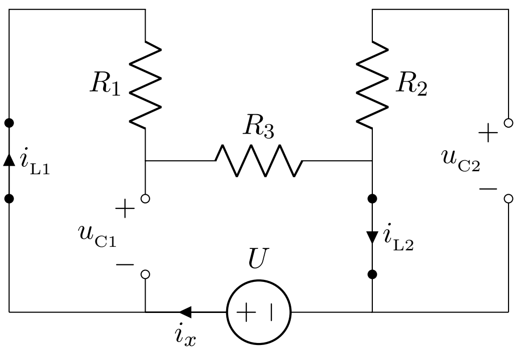
In this case it turns out useful to expand some parentheses, to reach eventually a simpler form,

$$u_x = \frac{R_1R_2 + R_3R_2 - R_1R_2 + R_1R_3 + R_3^2 - R_3\left(R_1+R_2+R_3\right)}{\left(R_1+R_3\right)\left(R_2+R_3\right)}U \quad \ldots = 0!$$

This may seem surprising at first, but now that we've got the idea of "what makes u_x zero" we might notice an explanation that could have been a big short-cut if noticed earlier! The inductor L_2 is, by continuity, carrying the same current as before the switch opened. In order for this to be true, the currents in R_2 and R_3 should sum to the same value as before. The nodes at the other ends of those resistors have, due to continuity of voltage in the two capacitors, kept the same potentials as before, relative to the node below L_2. Therefore, in order to keep their total current unchanged into L_2, the voltage u_x must be the same as before, which is zero on account of the behaviour of an inductor in equilibrium.

$$\text{Solution of subquestion } \mathbf{c)} \quad P_{\mathrm{L2}} = -u_x \cdot i_{\mathrm{L2}} = 0.$$

The end of time ... $t \to \infty$

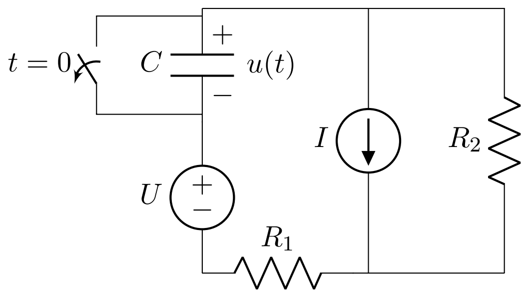


This is another equilibrium. The only difference from $t=0^-$ is that the switch is now an open-circuit. That makes it simpler, as R_2 now has one end left open, so U, R_1, R_3 is the only loop where current can flow.

The current out from the $+$-terminal of source U is therefore just $i_x = \frac{U}{R_1+R_3}$. The power it supplies to the circuit is Ui_x.

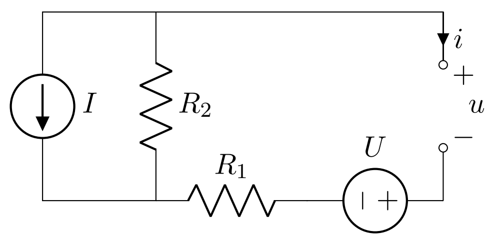
Solution of subquestion **d)** $\quad P_{\rm U} = Ui_x = \dfrac{U^2}{R_1+R_3}.$

Q2.



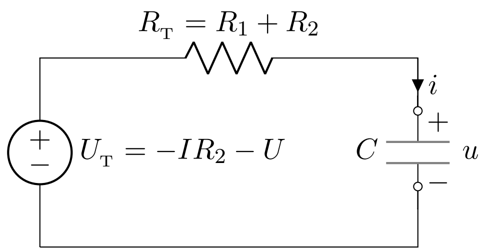
Again (boringly) there's only one change over all time. We care about $t>0$. Often in such a case we need to solve anyway at $t=0^-$ in order to find the initial conditions. In this case we're lucky: the closed switch ensures zero voltage on the capacitor, so no further solution of the initial state is needed.

Two-terminal equivalent, then three parameters.



The circuit to the right is what is 'seen' by the capacitor for $t>0$. If the capacitor is connected between the marked terminals at $t=0$, with no initial charge, its current and voltage will have the same functions for $t>0$ as they would in the original circuit.

This circuit has been re-drawn to be in a more familiar form for two-terminal equivalents, with its terminals at the right.



At the right is the Thevenin equivalent of the circuit above it, with the capacitor C connected to the terminals. The values of $u(t)$ and $i(t)$ for $t>0$ should be the same as in the original circuit, if the initial condition of the capacitor is correctly set.

The Thevenin voltage (open-circuit voltage) can be found by first KCL to show that all I passes through R_2 in open-circuit conditions, and then KVL around the loop of u, U, R_1, R_2.

The Thevenin resistance can be found by setting independent sources to zero and simplifying the remaining resistors, as there is no dependent source.

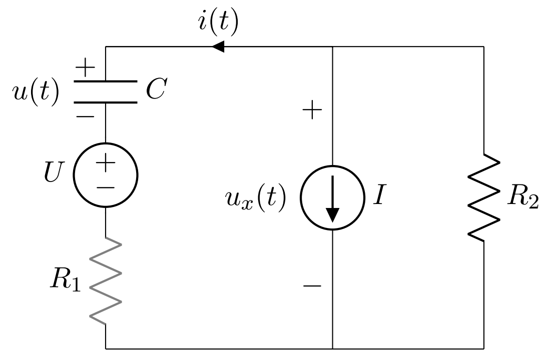
We know the initial condition $u(0)=0$, and that the final condition is $u(\infty) = U_{\rm T} = -IR_2 - U$.
The time-constant is $\tau = R_{\rm T}C = (R_1+R_2)\,C$.
Therefore,

$$u(t) \;=\; u(\infty) + \big(u(0)-u(\infty)\big)\mathrm{e}^{-t/C(R_1+R_2)} \;=\; -\,(IR_2+U)\left(1-\mathrm{e}^{-t/C(R_1+R_2)}\right) \qquad (t>0).$$

That's not the end, as the aim was to find the power from the source I.

The diagram on the right shows the original circuit in the state with the switch open ($t > 0$).

The marked voltage $u_x(t)$ is the voltage across the source I. This voltage is marked with "passive convention" relative to the current I, such that the current enters the + of the voltage. The power *out* of the source I is therefore $-Iu_x(t)$.



In order to find $u_x(t)$ we can write KCL at the top node. The current $i(t)$ in the left branch can be found without considering U or R_1, as it is linked to the now-known capacitor voltage $u(t)$ by $i = C\frac{\mathrm{d}u(t)}{\mathrm{d}t}$. (That does not mean that $i(t)$ is independent of U or R_1! The expression for $u(t)$ already includes those quantities, so they *can* appear in the final expression.)

$$i(t) + I + \frac{u_x(t)}{R_2} = 0 \quad \Longrightarrow \quad C\frac{\mathrm{d}u(t)}{\mathrm{d}t} + I + \frac{u_x(t)}{R_2} = 0.$$

$$u_x(t) = -R_2C\frac{\mathrm{d}u(t)}{\mathrm{d}t} - R_2I.$$

Substituting the earlier expression for $u(t)$,

$$u_x(t) \;=\; -R_2C\frac{\mathrm{d}}{\mathrm{d}t}\left\{ (-IR_2 - U)\left(1 - \mathrm{e}^{-t/C(R_1+R_2)}\right)\right\} - R_2I.$$

$$u_x(t) \;=\; -R_2C\frac{-1}{C\,(R_1+R_2)}\,(-IR_2 - U)\left(-\mathrm{e}^{-t/C(R_1+R_2)}\right) - R_2I.$$

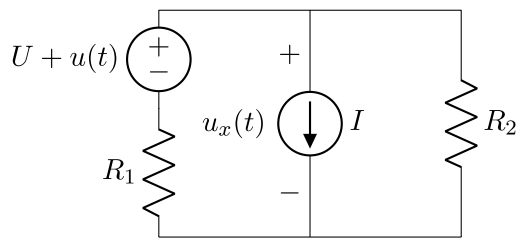
$$u_x(t) \;=\; \frac{R_2\,(U + IR_2)}{R_1+R_2}\mathrm{e}^{-t/C(R_1+R_2)} - R_2I.$$

And so the final answer is,

$$P_{\mathrm{I}} \;=\; -u_x(t)I \;=\; I^2R_2 - \frac{IR_2\,(U+IR_2)}{R_1+R_2}\mathrm{e}^{-t/C(R_1+R_2)}.$$

Alternative way to find u_x from u.

An alternative way to find $u_x(t)$ from $u(t)$ is to represent the left branch of the original circuit as a total voltage and a resistance, as shown on the right. An expression for u_x can be found by one KCL for the three branches in the above.



$$\frac{u_x(t) - U - u(t)}{R_1} + I + \frac{u_x(t)}{R_2} = 0.$$

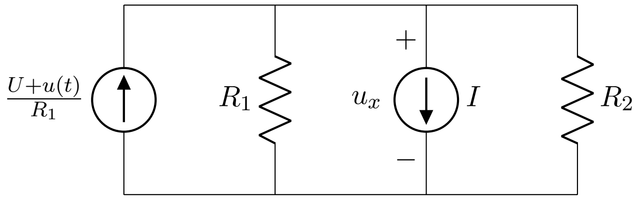
Note that it would *not* be valid to keep the function $u(t)$ if anything were changed in the circuit. We can represent the capacitor as $u(t)$ because that is the known solution for the capacitor *when it's connected to this particular circuit.*

Rearranging to find $u_x(t)$,

$$u_x(t)\frac{R_1+R_2}{R_1R_2} = \frac{U}{R_1} + \frac{u(t)}{R_1} - I,$$

$$u_x(t) \;=\; \frac{R_1 R_2}{R_1+R_2}\left(\frac{U}{R_1}+\frac{u(t)}{R_1}-I\right).$$

Another way to arrive at the above equation is source-transformation, as shown on the right, which makes it easy to write u_x as the product of the total current and the equivalent resistance,



$$u_x(t) \;=\; \frac{R_1 R_2}{R_1+R_2}\left(\frac{U+u(t)}{R_1}-I\right).$$

In either case, it can be simplified to

$$u_x(t) \;=\; \frac{R_2}{R_1+R_2}\big(U+u(t)-IR_1\big),$$

and the expression for $u(t)$ substituted into it to give

$$u_x(t) \;=\; \frac{R_2}{R_1+R_2}\left(U-IR_1-(IR_2+U)\left(1-\mathrm{e}^{-t/C(R_1+R_2)}\right)\right)$$

$$u_x(t) \;=\; -IR_2+\frac{R_2\,(U+IR_2)}{R_1+R_2}\,\mathrm{e}^{-t/C(R_1+R_2)} \quad (t>0).$$

Alternative ('risky') way to find $u_x(t)$ directly.

Our usual advice is to do as above: find the continuous quantity first, then find other quantities from that. However, the principle of initial value, final value and time-constant should be true for other current or voltage quantities, as long as we are careful to take the initial value as being at $t=0^+$, not $t=0^-$: for some quantities these need not be the same.

In this particular circuit, the opening of the switch has no initial effect on the circuit outside the parallel pair of the capacitor and switch. That's because the capacitor is connected directly across the switch and therefore forces the zero voltage to persist at $t=0^+$. (What *does* change is that now the current through the left branch of the whole circuit now passes through the capacitor instead of the switch, causing a non-zero $\mathrm{d}u/\mathrm{d}t$ and thus causing the capacitor voltage to change over time. At $t=0^+$ there's been too little time to see a change.

Based on the above, the only quantities in the circuit that will differ between $t=0^-$ and $t=0^+$ are the currents in the switch and capacitor. The voltage and total current in their branch remains the same, so the current source will not yet have 'seen' any change in the circuit. Therefore, the voltage across the current source will be the same at $t=0^+$ as at $t=0^-$, so we can find it from the equilibrium without having to do a further solution at $t=0^+$.

Define the current-source voltage as u_x, as shown in an earlier diagram. Use the solution found earlier for u_x in terms of u, but set $u=0$ which is appropriate for the situation at $t=0$. Thus,

$$u_x(0^+) \;=\; \frac{R_1 R_2}{R_1+R_2}\left(\frac{U+0}{R_1}-I\right) \;=\; \frac{R_2\,(U-IR_1)}{R_1+R_2}.$$

As $t\to\infty$, the whole left branch becomes open-circuit, so all the current I must pass through R_2,

$$u_x(\infty) \;=\; -IR_2.$$

The time-constant can be found in the same way as earlier, by finding the Thevenin resistance of the circuit that's connected to the capacitor at $t>0$, i.e. $R_{\mathrm{T}}=R_1+R_2$, from which $\tau=C\,(R_1+R_2)$.

Putting these three things together,

$$u_x(t) \;=\; u_x(\infty)+\big(u_x(0^+)-u_x(\infty)\big)\,\mathrm{e}^{-t/C(R_1+R_2)} \;=\; -IR_2+\left(\frac{R_2\,(U-IR_1)}{R_1+R_2}+IR_2\right)\mathrm{e}^{-t/C(R_1+R_2)}.$$

$$u_x(t) \;=\; -IR_2+\frac{R_2\,(U+IR_2)}{R_1+R_2}\,\mathrm{e}^{-t/C(R_1+R_2)} \quad (t>0).$$